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A STUDY OF SOLAR MOTION BY HARMONIC ANALYSIS

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ABSTRACT

Determination of solar motion.—A method of obtaining the solar apex and velocity by graphical-harmonic analysis is here presented, which, it is claimed, excels Airy's least-squares method in its simplicity, ease, and adaptability to available data. The theory of the analysis is discussed and its further possibilities in the study of stellar parallaxes and group motion are pointed out.

To demonstrate the method, the radial motion of 476 stars was obtained from the publication of the *Dominion Astrophysical Observatory*, 2, No. 1, and their corresponding proper motion, from Boss's *Preliminary General Catalogue.* From these, the solar apex was determined with the following results:

$$\alpha=272^\circ\ ,\quad \delta=33^\circ\!.7\ ,\quad V=24.8\text{ km/sec.}\ ,\quad \pi_{av}=0.0092\ .$$

To show the relative accuracy of the harmonic analysis and the least-squares analysis, the radial velocities of sixty stars, of declination about 60°, were used; by harmonic analysis the apex and velocity were found to be

$$\alpha=266^\circ\ ,\qquad \delta=41^\circ.5\ ,\qquad V=20.5\text{ km/sec.}\ ;$$

by least squares,

$$\alpha=256^\circ\ ,\qquad \delta=60^\circ\ ,\qquad V=17.3\text{ km/sec.}\ ,$$

the former being nearer to the values obtained from the most recent and extensive studies of the determination of the solar apex.

Concerning the ether-drift experiment.—Inasmuch as the foregoing method is the outcome of the investigation of the recent data on the ether-drift experiment (*Science*, April 30, 1926), a discussion of the latter is given.

INTRODUCTION

Nearly all of the recent determinations of the probable position of the apex and the speed of the solar motion have employed Airy's least-squares method.

It has been pointed out by a number of writers that if, in the determination of the position of the solar apex, the stars to be used are grouped in small areas, the majority of the stars in one area play a small part in the result, since the presence of a few large proper motions virtually determines the mean proper motion of the area.[1]

The nature of the process of the least-squares solution does not enable the investigator to know readily and easily what effect a certain star or a group of stars will produce in the result. It might be added that the method involves considerable labor, which limits the extent of the investigation.

In this paper we propose a method free from the objectionable features of Airy's least-squares solution; it is amply accurate, considering the precision of the data available, and is much less laborious; it provides a simple method for the study of the group motion of stars and of stellar parallaxes.

THEORETICAL CONSIDERATIONS

It is assumed here that the peculiar motions of stars are at random.

Denote by α_a and δ_a the right ascension and declination of the solar motion, and by V its speed (in kilometers per second). Then the angular velocity of the solar motion, V'' (seconds of arc per year), will be equal to $V\pi_a/4.74$, when π_a is the average parallax of the stars used in the determination. Also denote the proper motion in right ascension and declination of a star whose right ascension is α and declination δ by μ_α and μ_δ (measured in seconds of arc per year), its radial velocity by ρ (kilometers per second) and its parallax by π (seconds of arc).

Assuming the directions of ρ, μ_α, and μ_δ through the star as the axes of a system of reference, we have from trigonometry:

$$-\rho = V \cos \delta_a \cos \delta \cos (\alpha_a - \alpha) + V \sin \delta_a \sin \delta \ , \tag{1}$$

$$-\mu_\alpha \cos \delta = V'' \cos \delta_a \sin (\alpha_a - \alpha) \ , \tag{2}$$

$$-\mu_\delta = -V'' \cos \delta_a \sin \delta \cos (\alpha_a - \alpha) + V'' \sin \delta_a \cos \delta \ . \tag{3}$$

Let δ, the declination of the star, be constant; that is, consider the motions of a band of stars of the same declination but having

[1] A. S. Eddington, *Stellar Movements and the Structure of the Universe*, p. 80.

any value of right ascension, α. Then, since V, V'', δ, δ_a, and α_a are constants, it is seen that $\mu_\alpha \cos\delta$, μ_δ, and ρ are simple harmonic functions of α.

If values of ρ are plotted against α, that is, if the radial velocity of each star is plotted as ordinate and its right ascension as abscissa, then an approximate sine curve will be formed of the type $\rho = P_\rho + A_\rho \sin(\alpha + \phi_\rho)$, where P_ρ is the "average ordinate" or the displacement of the axis of the curve from the zero line, A_ρ is the amplitude of the curve, and ϕ_ρ its phase. These three quantities are represented by:

$$P_\rho = -V \sin\delta_a \sin\delta\ , \tag{4}$$

$$A_\rho = V \cos\delta_a \cos\delta\ , \tag{5}$$

$$\phi_\rho = 270^\circ - \alpha_a\ . \tag{6}$$

Similarly, if the curves for $\mu_\alpha \cos\delta$ and μ_δ are drawn, their average ordinates, amplitudes, and phases are:

$$P_{\mu_\alpha} = 0\ , \tag{7}$$

$$A_{\mu_\alpha} = V'' \cos\delta_a\ , \tag{8}$$

$$\phi_{\mu_\alpha} = -\alpha_a \tag{9}$$

$$P_{\mu_\delta} = -V'' \sin\delta_a \cos\delta\ , \tag{10}$$

$$A_{\mu_\delta} = V'' \cos\delta_a \sin\delta\ , \tag{11}$$

$$\phi_{\mu_\delta} = 90^\circ - \alpha_a\ . \tag{12}$$

If, instead of δ, α is made constant; that is, if, instead of using a declination band of stars, a meridian band is used, then from equations (1) and (3) it can be seen that ρ and μ_δ are simple harmonic functions of δ. If curves are plotted, the average ordinates, amplitudes, and phases are, in this case:

$$P'_\rho = 0\ , \tag{13}$$

$$A'_\rho = V\sqrt{\sin^2\delta_a + \cos^2\delta_a \cos^2(\alpha - \alpha_a)}\ , \tag{14}$$

$$\phi'_\rho = \tan^{-1}[\cot\delta_a \cos(\alpha - \alpha_a)]\ , \tag{15}$$

$$P'_{\mu_\delta} = 0 \tag{16}$$

$$A'_{\mu_\delta} = V''\sqrt{\sin^2\delta_a + \cos^2\delta_a \cos^2(\alpha - \alpha_a)}\ , \tag{17}$$

$$\phi'_{\mu_\delta} = \tan^{-1}[\tan\delta_a \sec(\alpha - \alpha_a)]\ . \tag{18}$$

The corresponding meridian band from equation (2) will be a straight line displaced from the axis by:

$$P'_{\mu_a} = V'' \cos \delta_a \sin (\alpha - \alpha_a) . \tag{19}$$

METHOD OF ANALYSIS

In the actual analysis three curves are drawn with right ascension as abscissa and with ρ, $\mu_\alpha \cos \delta$, and μ_δ, respectively, as ordinates. The stars are grouped in declination bands 10° wide. Corresponding curves for meridian bands one hour wide are drawn. In practice the scale used was 400 mm for twenty-four hours of right ascension, 2 mm for 1 km per second radial velocity, 2 mm for 0″.0015 proper motion in right ascension, and 1.5 mm for 0″.001 proper motion in declination.

The average ordinate can be found by obtaining the area in square millimeters, with a planimeter, between the curve and the zero line, from zero to twenty-four hours, and dividing the area by 400 mm, the length of the base.

The phase and amplitude are obtained by means of the Henrici harmonic analyzer,[1] which is a device for finding the coefficients of the Fourier series:

$$y = A_1 \sin (\theta + \phi_1) + A_2 \sin (2\theta + \phi_2) + A_3 \sin (3\theta + \phi_3) + \dots\dots$$

It is necessary to use only the integrators for the first component, since the theoretical curve is a simple sine curve.[2]

ADVANTAGES OF METHOD

It is possible to plot and analyze the three curves (equations [1], [2], and [3]) for a declination band of one hundred stars in one afternoon. The proper and radial motions are at all times visible on the curves, and discrepancies can be noted and errors corrected quickly and easily. If a separate determination for a group of stars is desired, this can readily be accomplished by joining the points corresponding to these stars by straight lines and analyzing the

[1] O. Henrici, "On a New Harmonic Analysis," *Philosophical Magazine*, Vol. 38, 1894.

[2] D. C. Miller, "The Henrici Harmonic Analyzer," *Journal of the Franklin Institute*, September, 1916.

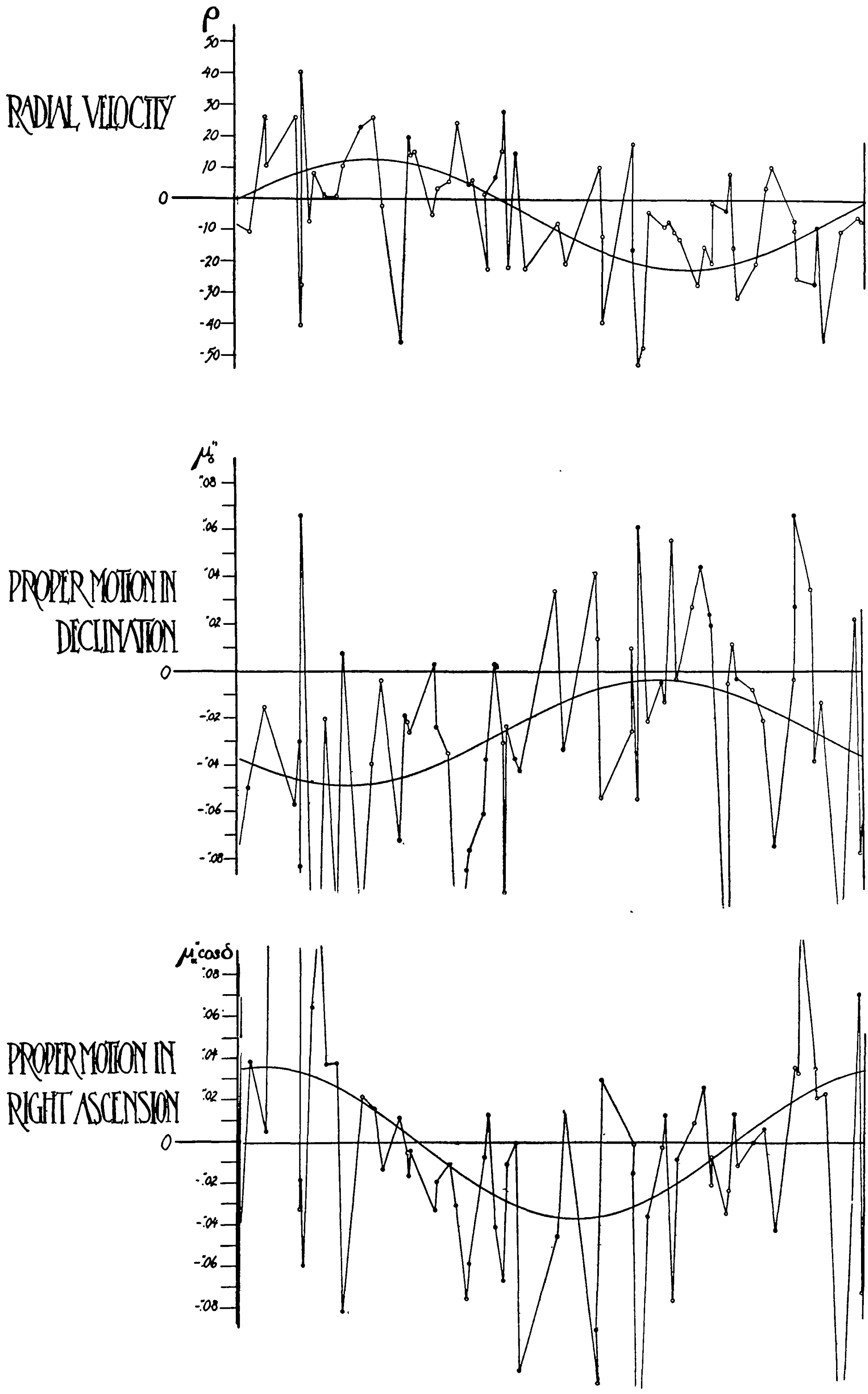

FIG. 1.—Declination band, 25°–35°. The smooth sine curves have the phase, amplitude, and average ordinate obtained by harmonic analysis of the irregular data curves.

resulting curve, a matter of a few minutes' time. Also, if it is desired to note the effect of omitting any star in the determination, it is evident that it can easily be accomplished.

Large discrepancies, which are always present in the data available and which greatly affect the final result when the least-squares solution is employed, have very little effect when the harmonic analyzer is used, especially when the stars are close together in right ascension.

If a correction in the form of an additive constant, or of a sine or cosine function of α, is to be applied to the data, as is the case with stars from Boss's *Catalogue*, such a correction can be applied to the results of the analyzed curve, without the need of correcting each individual proper motion or radial velocity.

Some irregularities in the weights of individual observations are likely to occur if the spacing in right ascension is irregular, but a judicious averaging of points can be made which will obviate this to some extent.

OTHER APPLICATIONS OF THE METHOD

The determination of the solar apex is but one of the results which can be obtained from these curves by this method of analysis. Other applications are here suggested.

Draw over each data curve the sine curve representing the determined solar motion for that band. The vertical distance between this sine curve and the points representing the motion of an individual star denote by Δ_ρ, Δ_{μ_α}, and Δ_{μ_δ}; Δ_ρ indicating radial motion, Δ_{μ_α} and Δ_{μ_δ} indicating proper motion in right ascension and declination, respectively, and being considered *plus* when above the sine curve and *minus* when below. Thus we are able to get a set of three curves for each declination band from which a general idea of group motion can easily be obtained. Stars comparatively near together, whose Δ_ρ are nearly equal and of the same sign, and whose Δ_{μ_α} and Δ_{μ_δ} are of the same sign, may be considered as traveling in a group.

If the parallax of one star in a group is known, then the ratio between the Δ_μ of this star and the Δ_μ of another star of unknown parallax in the same group will be equal to the ratio of their parallaxes. This is, of course, possible only when observational errors are small.

If the $\mu_\alpha \cos \delta$ and μ_δ curves are analyzed, the solar velocity will be found in seconds of arc, as V''. In order to reduce this to kilometers per second, the average parallax of the stars must be known. To obtain this, advantage is taken of the properties of the ρ and μ_δ curves of any meridian band. The amplitudes of each, from equations (14) and (17), are in the ratio of $V : V''$.

The curves, in their simplicity of form and ease of analysis, show many further possibilities which cannot be developed here, but which will no doubt be of considerable help in the study of stellar motions.

SOLAR MOTION DERIVED FROM 476 STARS

Material used.—The material consists of those stars in the *Publications of the Dominion Astrophysical Observatory*[1] whose radial velocities do not exceed 50 km/sec. after being corrected for the solar motion, and whose proper motions do not exceed 0."5 per year; in all, 476 stars were used. The proper motions of the stars were obtained from Boss's *Preliminary General Catalogue* and were corrected as described on page xxviii of this *Catalogue*. Further corrections were applied to the proper motions in declination as recently proposed by Raymond.[2]

Method of computation.—The stars were arranged in declination bands lying between 0° and 5°, 5° and 15°, 15° and 25°, 25° and 35°, 35° and 45°, 45° and 55°, and 55° and 65°, giving seven bands whose average declinations were $2\frac{1}{2}°$, 10°, 20°, 30°, 40°, 50°, and 60°.

As the accuracy of the right ascension determined by this method depends on the amplitude of the curve, the amplitude of the curve was used as weight for each individual determination.

Since the radial velocities contain a constant error K_ρ which would displace the curve vertically, and which, therefore, would appear as a constant error in the P_ρ's, the results of the ρ curves were solved to obtain K_ρ, and the values were then corrected.

[1] "The Radial Velocities of 594 Stars," *op. cit.*, **2**, No. 1

[2] H. Raymond, "The Correction to the Declination System of Boss's *Preliminary General Catalogue*," *Astronomical Journal*, **36**, 17–18, 1925.

In order properly to weight the determination of δ_a, equations of the following form were used:

$$V \sin \delta_a = \frac{\Sigma P_\rho}{\Sigma \sin \delta},$$

$$V \cos \delta_a = \frac{\Sigma A_\rho}{\Sigma \cos \delta}.$$

TABLE I

DATA: DECLINATION BANDS

Declination Bands	A_ρ	P_ρ	P_ρ Corrected	ϕ_ρ	a_ρ
0°	20.0	− 0.1	1.5	345°	285°
10	22.3	− 0.5	1.1	19	251
20	21.5	+ 1.4	3.0	344	286
30	15.8	4.4	6.0	10	260
40	18.7	7.1	8.7	351	279
50	15.3	8.0	9.6	360	270
60	7.7	+11.8	13.4	4	266

TABLE II

DATA: DECLINATION BANDS

Declination Bands	A_{μ_a}	ϕ_{μ_a}	a_{μ_a}	A_{μ_δ}	P_{μ_δ}	ϕ_{μ_δ}	a_{μ_δ}
0°	0.0487	81°	279°	0.0036	0.0284	182°	268°
10	0.0409	60	300	0.0061	0.0236	196	254
20	0.0332	69	291	0.0108	0.0268	168	282
30	0.0349	81	279	0.0230	0.0258	209	241
40	0.0501	90	270	0.0331	0.0315	171	279
50	0.0266	74	286	0.0306	0.0106	194	256
60	0.0342	72	288	0.0180	0.0097	186	264

Results.—Using the material and the method of harmonic analysis as described, the results for the apex and velocity of the solar motion are as follows:

From radial motions alone:

$$a_a = 270^\circ 45',$$

$$\delta_a = 31^\circ 45',$$

$$V = 25.1 \text{ km/sec.}$$

From proper motions alone:

$$\alpha_a = 273^\circ 00' \,,$$
$$\delta_a = 35^\circ 40' \,,$$
$$V'' = 0\overset{''}{.}0473 \,.$$

From radial motions and proper motions combined:

$$\alpha_a = 272^\circ 00' \,,$$
$$\delta_a = 33^\circ 40' \,,$$
$$V = 24.8 \text{ km/sec.} \,,$$
$$\pi_{av} = 0\overset{''}{.}0092 \,.$$

Two sets of meridian curves were drawn, one for 0° and 180°, and the other for 90° and 270°. Each band was 15° wide. The analysis gave $A'_\rho / A'_{\mu_\delta} = V/V'' = 580$, that is, $\pi_{av} = 0\overset{''}{.}0082$.

To show the relative accuracy of the harmonic analysis and the least-squares analysis, the ρ curve for the 60° declination band, consisting of sixty stars, was used, giving by the method of harmonic analysis:

$$\alpha_a = 266^\circ \,, \qquad \delta_a = 41\overset{\circ}{.}5 \,, \qquad V = 20.5 \text{ km/sec.} \,;$$

while the least-squares method applied to the same set of stars gives:

$$\alpha_a = 256^\circ \,, \qquad \delta_a = 60^\circ \,, \qquad V = 17.3 \text{ km/sec.}$$

The most recent and extensive determination of the solar apex, made by Dr. Ralph E. Wilson,[1] gives:

$$\alpha_a = 270\overset{\circ}{.}8 \,, \qquad \delta_a = 27\overset{\circ}{.}1 \,, \qquad V = 19.0 \text{ km/sec.} \,,$$

from which it is seen that the treatment of a group of observations by harmonic analysis gives a result nearer to the values obtained from this study, the most complete which has been made.

[1] "The Solar Motion Problem," *ibid.*

SOME THEORETICAL CONSIDERATIONS OF THE SOLAR MOTION AS DETERMINED BY ETHER-DRIFT

The foregoing investigation is the outcome of the theoretical consideration of the Michelson and Morley ether-drift experiment, as it has been undertaken recently by Professor Dayton C. Miller.[1] The formulae used in the determination of the solar motion relative to the ether, from observations made by the interferometer, can be derived as follows:

The interferometer measures the projection of the apparent relative motion of the earth and of the ether, giving the azimuth A and magnitude v of this projection as functions of the sidereal time. If α and δ represent the right ascension and declination of the relative motion of the earth and the ether, and V its velocity, we have the familiar equations:

$$\cos z = \sin\delta \sin\phi + \cos\delta \cos\phi \cos t\ , \tag{20}$$

$$\sin z \cos A = +\sin\delta \cos\phi - \cos\delta \sin\phi \cos t\ , \tag{21}$$

$$\sin z \sin A = \cos\delta \sin t\ , \tag{22}$$

$$v = V \sin z\ , \tag{23}$$

$$t = \theta - \alpha\ , \tag{24}$$

where ϕ is the latitude of the observatory, z is the zenith distance of the apex, and θ is the sidereal time of the observation.

As the observed azimuth and magnitude of the drift are quite independent of each other, each gives a separate determination of the apex.

Solar apex derived from magnitude of ether-drift.—If V and its direction are assumed to be constant throughout a short period of time (about ten days), and ϕ and δ are positive, equations (20) and (23) show that v is a minimum when $t=0$. Therefore we may write:

$$\left.\begin{aligned} \alpha &= \theta \text{ when } v \text{ is minimum,} \\ z_{min.} &= |\,\phi - \delta\,| \text{ when } v \text{ is minimum}\ , \\ v_{min.} &= V \sin |\,\phi - \delta\,|\ . \end{aligned}\right\} \tag{25}$$

[1] "Significance of the Ether-Drift Experiment of 1925 at Mount Wilson," *Science*, April 30, 1926.

If $\delta \leqq 90° - \phi$, the maximum value of v occurs when $z = 90°$, and therefore

$$v_{max.} = V \tag{26}$$

so that

$$\frac{v_{min.}}{v_{max.}} = \sin \mid \phi - \delta \mid$$

or

$$\delta = \phi \pm \sin^{-1}\left(\frac{v_{min.}}{v_{max.}}\right) . \tag{27}$$

If $\delta > 90° - \phi$, a rough value of δ may be obtained from equation (27), or the maximum value of v occurs when $z = 180° - (\phi + \delta)$, from which we obtain:

$$\frac{v_{min.}}{v_{max.}} = \frac{\sin \mid \phi - \delta \mid}{\sin (\phi + \delta)} . \tag{28}$$

From the foregoing we observe that if v is plotted against sidereal time, equations (23), (25), (26), (27), and (28) will determine the direction and velocity of the solar system relative to the ether.

The theoretical curve for v is

$$v = V \sin [\cos^{-1} (\sin \delta \sin \phi + \cos \delta \cos \phi \cos t)] . \tag{29}$$

Applying equations (25) and (28) to the observed magnitude of ether-drift (average for April, August, and September, 1925, and February, 1926) as observed by Professor Miller, the direction of the apex was found to be:

$$\alpha = 252° , \qquad \delta = 71° ,$$

Solar apex derived from azimuth of ether-drift.—The azimuth A is measured from the north point, *plus* toward the east and *minus* toward the west. Since the interferometer observations cannot distinguish between azimuths 180° apart, we shall consider A numerically less than 90°.

CASE I, $\delta \geqq \phi$

It is clear that when $t=0$, v is minimum and $A=0^\circ$. As t increases, A becomes negative, and when t becomes equal to twelve hours, A becomes zero again. For greater values of t, the azimuth becomes positive. Hence when the azimuth passes from east to west t becomes equal to 0, or $\theta=\alpha$, which corresponds to the place where the curve of azimuth against time crosses the time-axis, when the azimuth passes from a maximum to a minimum.

Dividing equation (21) by equation (22) we obtain

$$A=\cot^{-1}[-\sin\phi\cot t+\tan\delta\cos\phi\csc t]\ . \qquad (30)$$

The maximum value of A is such that

$$\sin A_{max}=\cos\delta\sec\phi\ .$$

This gives the required value of δ.

The average observed azimuth for the four epochs (April, August, and September, 1925, and February, 1926) has for an X-axis (time-axis) a line 62° below the X-axis of equation (30).

It was found necessary, therefore, to shift the axis of the observed azimuth 62° (*Science*, April 30, 1926, p. 10).

With this modification and from the foregoing discussion, the direction of the apex becomes:

$$\alpha=255^\circ\ , \qquad \delta=70^\circ\ .$$

The curves for magnitude and azimuth (equations [29] and [30]), while they are not simple sine curves, may be considered as harmonic for not more than three components of the Fourier series. These components were found, and the curves were synthesized.[1] Thus α, δ, and V were computed.

CASE II, $\delta<\phi$

Here again, when $t=0$, $A=0^\circ$. As t increases, A receives the following successive values:

$$0\ ,\ \ldots\ ,\ 90^\circ\ ,\ -90^\circ\ ,\ \ldots\ ,\ 0^\circ\ ,\ \ldots\ ,\ 90^\circ\ ,\ -90^\circ\ ,\ \ldots\ ,\ 0^\circ\ .$$

Hence when $A=0^\circ$, $\alpha=\theta$ or $\alpha=\theta+12^h$.

[1] D. C. Miller, "A 32-Element Harmonic Synthesizer," *Journal of the Franklin Institute* (January, 1916), pp. 51–81.

The value of t when $A = 90°$ being known, equation (30), will give δ.

It is also possible to compute the north-south and east-west components of the magnitude, and thus obtain two sets of curves which correspond respectively to the μ_a and μ_δ curves (equations [2] and [3]) of the first section of this paper. Their analysis can be made in much the same manner as that of the proper-motion curves.

The writers are under obligation to Professor Dayton C. Miller for his kindness in making available the apparatus for the harmonic analysis and his inspiration and help in carrying on the work.

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