

On the Possibility of Terrestrial Tests of the General Theory of Relativity.

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The recognition of the constant c as a truly universal constant is intimately connected with the development of the theory of relativity. In fact, the constancy of the velocity of light in vacuo for any system of inertia is a consequence of and a necessary condition for the validity of the principle of relativity itself. At this International Congress of Fundamental Constants, it may therefore be pertinent to review the experimental basis of this theory and to discuss the possibilities of new experimental verifications of relativity effects.

During the last fifty years, a large number of consequences of the special theory of relativity have been strikingly verified by experiments. Nevertheless, the experiments which give a direct verification of the *principle of relativity* of sufficient accuracy are limited to the two classical experiments by MICHELSON and IVES. It has been pointed out previously [1] that a slightly generalized Michelson experiment, where the interferometer is filled with a strongly refractive medium, would supply a further proof of the principle of relativity, which could not be explained away by an ad hoc hypothesis regarding the contraction of moving bodies. However, in view of the overwhelming experimental evidence in favour of the special theory, it seems hardly worthwhile to go through the trouble of repeating this rather difficult and costly experiment. It is therefore interesting to note that, as we shall see, some of the very accurate time measuring instruments constructed in recent years can be used for a new independent direct check of the principle of relativity.

In contrast to the special theory, the general theory of relativity cannot be considered satisfactorily verified by experiments. In spite of the fundamental difference in principles between Einstein's and Newton's theories of

gravitation, only the effects of large masses like that of the sun could so far give rise to measurable differences in the predictions of the two theories. Therefore, the three classical Einstein effects—the advance of the perihelion of Mercury, the gravitational shift of spectral lines, and the deflection of light in the gravitational field of the sun—have up to now been the only experimental tests of the theory, encumbered with all the uncertainties inherent in such astronomical measurements. Lately, this situation has radically changed due to the enormous development of experimental technique and the increased economic resources available for physical experiments. Although the gravitational effects on the earth are much smaller than in the space around the sun, it seems possible in a near future to obtain new tests of the general theory of relativity by terrestrial experiments which have the advantage over astronomical measurements of well defined experimental conditions. In the following, we shall discuss these possibilities.

The idea which suggests itself is an attempt at using « atomic clocks » to verify the relativistic formula for the rate of clocks placed at different potentials in a gravitational field. In our discussion, we shall confine ourselves to one type of atomic clocks, the « maser » [2], in which ammonia molecules are used as balance of the clock. The device utilizes a focused beam of excited ammonia molecules entering a cavity. For sufficiently high beam strength, the molecules can maintain a very monochromatic oscillation in the cavity due to microwave transitions in the molecules. The frequency of this oscillation determines the rate of the clock which, over long periods, has been shown to be accurate to at least one part in 10^{10} . Over shorter periods, of the order of one second, the accuracy is even considerably higher, up to 10^{-12} . A comparison of the rate of two masers is made by counting the beats of interference between the characteristic frequencies of the two masers.

Since the molecules enter the cavity with a certain velocity u , the frequency ν emitted by the molecules will, on account of the Doppler effect, depend on the direction of emission. The rate of the maser, or the characteristic frequency ν_m , will now be a certain mean value of the Doppler frequency, i.e.

$$(1) \quad \nu_m = \int \nu(\mathbf{e})f(\mathbf{e}) d\Omega,$$

where \mathbf{e} is a unit vector in the direction of the photon emitted, and the weight function $f(\mathbf{e})$ depends on the geometrical arrangement of the apparatus [3].

Let us, to begin with, disregard the principle of relativity and calculate ν_m on the basis of the absolute ether theory. In this theory, we have approximately to the second order in the velocities [4]

$$(2) \quad \nu = \nu_0 \left[1 + \frac{\mathbf{e} \cdot \mathbf{u}}{c} + \frac{(\mathbf{e} \cdot \mathbf{u})^2}{c^2} + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2} \right],$$

where \mathbf{u} is the velocity of the molecules in the laboratory system, while \mathbf{v} is the absolute velocity of the laboratory. Further, ν_0 is the proper frequency for $\mathbf{u} = 0$. By introduction of (2) into (1), we get for the characteristic frequency

$$(3) \quad \nu_m = \nu_0 \left[1 + g(u) + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2} \right],$$

where $g(u)$ is the mean value of $\mathbf{e} \cdot \mathbf{u}/c + (\mathbf{e} \cdot \mathbf{u})^2/c^2$ and, thus, a function of the magnitude of \mathbf{u} only. However, on account of the scalar product term in (3), the rate of the maser should, according to (3), depend on its orientation with respect to the absolute velocity vector \mathbf{v} . In the actual maser, the majority of the molecules have a velocity $\mathbf{u} = 4 \cdot 10^4$ cm/s, and the absolute velocity \mathbf{v} of the laboratory may be put equal to the velocity of the earth in its orbit around the sun, i.e. $v = 3 \cdot 10^6$ cm/s. The relative change of the rate of the maser by a rotation through an angle of 180° should thus, according to the absolute ether theory, be of the order of 10^{-10} , an effect which it is quite possible to detect with the present accuracy of the maser. On account of the rotation of the earth around its axis, it is even not necessary to rotate the apparatus in order to detect the effect. If one uses two masers with opposite directions of the molecular beams, the effect in question should simply give rise to a diurnal variation of the relative rates of the two masers. On the other hand, according to the principle of equivalence and the principle of relativity, a rotation of the apparatus should *not* have any effect on the rate, as is seen at once if one introduces a local system of inertia at the space-time point in question for our description of the phenomena. Thus, if no diurnal variation of the rate is found, this may be regarded as a new check of the principle of relativity.

We shall now calculate the characteristic frequency of the maser according to the general theory of relativity. In an arbitrary gravitational field, we have instead of (2) the following exact formula for the Doppler effect (see Appendix A):

$$(4) \quad \nu = \nu_0 \frac{E_0}{E[1 - c'(\mathbf{p} \cdot \mathbf{e})/E]},$$

where E and \mathbf{p} are energy and momentum of the molecule in the gravitational field, and E_0 is the energy of the molecule at rest. Further,

$$(5) \quad c' = c\sqrt{1 + 2\chi/c^2},$$

where χ is the scalar gravitational potential, and ν_0 is the frequency emitted by a molecule at rest in the field. This frequency is connected with the proper

frequency $\hat{\nu}_0$ of the molecule when placed at rest in a system of inertia by the formula (Eq. (A.19) in Appendix A)

$$(6) \quad \nu_0 = \hat{\nu}_0 \sqrt{1 + 2\chi/c^2}.$$

Expanding E and p in powers of the velocity u , and keeping terms up to the second order only, gives

$$(7) \quad \nu = \nu_0 \left[1 - \frac{1}{2} \frac{u^2}{c'^2} + \frac{(\mathbf{u} \cdot \mathbf{e})^2}{c'^2} + \frac{\mathbf{u} \cdot \mathbf{e}}{c'} \left(1 + \frac{\boldsymbol{\gamma} \cdot \mathbf{u}}{c'} \right) \right],$$

which, by introduction into (1), gives the characteristic frequency ν_m . The exact value of ν_m will, of course, depend on the precise form of the weight function. If we assume that $f(\mathbf{e})$ is constant, independent of the direction, we get

$$(8) \quad \nu_m = \nu_0 \left[1 - \frac{1}{6} \frac{u^2}{c'^2} \right].$$

On the other hand, if $f(\mathbf{e})$ is assumed to be different from zero only for directions \mathbf{e} perpendicular to \mathbf{u} , we get

$$(9) \quad \nu_m = \nu_0 \left[1 - \frac{1}{2} \frac{u^2}{c'^2} \right].$$

In any case, the rate of the maser decreases with increasing energy of the molecules in the beam. This effect could in principle be checked by comparing the rate of two masers with different beam velocities, and this would effectively mean a new test of the relativistic Doppler formula. However, with the thermal velocities used so far, where $u = 4 \cdot 10^4$ cm/s at room temperature, the last terms in (8) and (9) are of the order of

$$(10) \quad \frac{u^2}{c^2} \approx 10^{-12},$$

which is perhaps somewhat too small to be measured with the present accuracy of the apparatus.

Neglecting terms of order $(u^2/c^2)((\chi_2 - \chi_1)/c^2)$, we get by (1), (6), and (7) for the ratio (τ_2/τ_1) of the rates of two identical masers placed at places with the potentials χ_2 and χ_1

$$(11) \quad \frac{\tau_2}{\tau_1} = \frac{\sqrt{1 + 2\chi_2/c^2}}{\sqrt{1 + 2\chi_1/c^2}} \approx 1 + (\chi_2 - \chi_1)/c^2.$$

If the two masers are connected by a cable or a wave guide, this effect can

in principle be measured by counting the beats between the characteristic frequencies of the two clocks. The formula (11) is in accordance with the general relativistic formula

$$(12) \quad d\tau = dt \sqrt{1 + 2\chi/c^2}$$

for the proper time $d\tau$ of an ideal standard clock at rest in a gravitational field as compared with the rate dt of the co-ordinate clocks. The equation (12), which is a simple consequence of the principle of equivalence, can also be derived by applying the laws of relativistic mechanics to the mechanism of a clock [5]. Under certain conditions, which are amply satisfied in the case of the oscillating ammonia molecules, (12) has been shown to be a consequence of the equations of motion of general relativity and in particular of the relativistic formula for the dependence of the rest mass m_0 of a particle on the gravitational potential

$$(13) \quad m_0 = \frac{\overset{\circ}{m}_0}{\sqrt{1 + 2\chi/c^2}}.$$

Here, $\overset{\circ}{m}_0$ is the mass of the particle when placed at rest in a system of inertia. An experimental verification of the equation (11) is therefore simultaneously a test of the relativistic formula (13) for the mass of a particle. For two masers, one at sea level, the other at a height h of 3 km, say, we get by (11) for the relative difference in rate

$$(14) \quad \Delta = \frac{\tau_2 - \tau_1}{\tau_1} = \frac{\chi_2 - \chi_1}{c^2} \approx \frac{|\text{grad } \chi| h}{c^2} \approx \frac{10^3 \cdot 3 \cdot 10^5}{9 \cdot 10^{20}} = \frac{1}{3} \cdot 10^{-12}.$$

Again, this effect is at the edge of what can be observed with the present accuracy of the available instruments, and we cannot gain much by climbing higher mountains.

Although it sounds somewhat fantastic at the moment, it may well be that, before clocks of considerably higher accuracy are constructed, it will be possible to use artificial satellites, in which case the effect in question can be made a thousand times bigger. As shown in Appendix B, we get for the relative difference in rate of two atomic clocks, one placed at the surface of the earth, the other in a satellite

$$(15) \quad \Delta = \frac{\tau_{\text{sat.}} - \tau_{\text{earth}}}{\tau_{\text{earth}}} = \frac{kM}{c^2} \left(\frac{1}{r} - \frac{3}{r_1 + r_2} \right),$$

where r , r_1 , and r_2 are the radius of the earth and the radii of the smallest and

largest distance of the satellite in its orbit around the earth, respectively. M is the mass of the earth and k is the gravitational constant. Hence,

$$(16) \quad \Delta = 0.7 \cdot 10^{-9} \cdot \left(1 - \frac{3r}{r_1 + r_2} \right).$$

The case of a circular orbit has already been discussed in detail by Dr. S. F. SINGER [6], who shows that the effect should be perfectly well observable by means of present day radio technique. The main point in SINGER's proposal is to use a counting method instead of a comparison of the frequencies of waves. Suppose the satellite clock has associated with it a scaler which counts its ticks and sends a short signal to the ground after each predetermined number of ticks. If the accumulation time or running time of the clock is sufficiently large, the detailed means of comparison between the satellite scaler and the ground clock becomes relatively unimportant. Also, if the errors are truly random, it is only a matter of time when the relativistic effect can be seen above the noise. According to (16) it will obviously be advantageous to use a highly eccentric satellite orbit $r_2 \gg r_1 \simeq r$, in which case the satellite clock has its maximum rate corresponding to $\Delta = + 0.7 \cdot 10^{-9}$. The communication signal could then, for instance, be sent each time the satellite has its closest distance to the earth.

The general relativistic effects considered so far may be regarded as consequences of general relativistic mechanics. Another characteristic action of the gravitational field is its influence on the velocity of propagation of radiation, which for instance is responsible for the deflection of light in a gravitational field. We shall now shortly discuss the possibility of measuring effects of this type by terrestrial experiments.

Any signal propagating through a medium at rest in a gravitational field, like that which exists in an earth-fixed system of reference, has a velocity w given by the general formula

$$(17) \quad w = \frac{c'}{(c/\tilde{w}) + \boldsymbol{\gamma} \cdot \mathbf{e}},$$

(see Appendix C). Here, \mathbf{e} is a unit vector in the direction of propagation of the radiation, and \tilde{w} is the velocity of the signal in a rest system of inertia for the medium in question. Further, c' is given by (5), and $\boldsymbol{\gamma}$ and χ are the vector and the scalar gravitational potential, respectively. For light going through a transparent medium of refractive index n , the velocity \tilde{w} is, for instance, c/n , and c' is equal to the velocity of light in vacuo for a direction \mathbf{e} perpendicular to the vector potential $\boldsymbol{\gamma}$.

The difficulty in checking the formula (17) by terrestrial experiments lies

in the fact that only differences in the velocity (17) for different space points give rise to observable effects. This follows at once from the principle of equivalence; for inside a region of essentially constant potentials, we may treat the phenomena used in the experiment from the point of view of a local system of inertia, where the gravitational effects disappear. It is therefore clear that the experimental arrangement must cover large areas. Further, there is in general a danger that uncontrollable variations in the properties of the medium (i.e. in \hat{w}) will overshadow the weak effects due to the gravitational field. There is one arrangement, however, in which this latter difficulty is eliminated. Consider two signals which, starting from the same point P , are going along a closed loop, but in opposite directions. The time intervals T_+ and T_- needed for the signals to make one turn are then, according to (17),

$$(18) \quad T_{\pm} = \oint_{(\pm)} \frac{d\sigma}{w} = \oint_{(+)} \frac{c d\sigma}{c' \hat{w}} \pm \oint_{(+)} \frac{\boldsymbol{\gamma} \cdot \mathbf{e}}{c'} d\sigma.$$

The time interval between the arrivals at P of the two signals after one turn is then completely independent of the properties of the medium traversed and equal to

$$(19) \quad \Delta t = 2 \oint_{(+)} \frac{\boldsymbol{\gamma} \cdot \mathbf{e}}{c'} d\sigma = \frac{2}{c} \oint_{(+)} a_{\sigma} d\sigma,$$

where the spatial vector \mathbf{a} is given by

$$(20) \quad \mathbf{a} = \boldsymbol{\gamma} / \sqrt{1 + 2\chi/c^2}$$

and $a_{\sigma} = \mathbf{a} \cdot \mathbf{e}$ is the component of \mathbf{a} in the direction of the line element $d\sigma$ along the curve. By means of Stokes' theorem, which holds also in a non-Euclidean space, (19) may be written as a two-dimensional integral over a surface Σ delimited by the closed path of the signals

$$(21) \quad \Delta t = \frac{2}{c} \int_{\Sigma} (\text{curl } \mathbf{a} \cdot \mathbf{n}) dA,$$

where \mathbf{n} is a unit vector in the direction of the normal to the surface element with the area dA .

We can now introduce an earth-fixed system of co-ordinates (x, y, z, t) in which the covariant components of the vector potential are

$$(22) \quad \gamma_i = (\gamma_x, \gamma_y, \gamma_z) = \left(-\frac{\omega y}{c'}, \frac{\omega x}{c'}, 0 \right)$$

(see Appendix C). Here, ω is the angular velocity of the earth around its axis. By a simple calculation, one gets (Appendix C, Eqs. (C.11)-(C.14))

$$(23) \quad \text{curl } \mathbf{a} = \boldsymbol{\Omega} + \boldsymbol{\Omega}_1,$$

where $\boldsymbol{\Omega}$ is a space vector lying in the direction of the axis of the earth and of the magnitude

$$(24) \quad |\boldsymbol{\Omega}| = \frac{2\omega}{c}.$$

Further $\boldsymbol{\Omega}_1$ is given by

$$(25) \quad \boldsymbol{\Omega}_1 = -\frac{2}{c^2}(\boldsymbol{\gamma}\mathbf{G}),$$

where

$$(26) \quad \mathbf{G} = -\text{grad } \chi$$

is the gravitational acceleration. Since $\boldsymbol{\gamma}$ is pointing to the east, and \mathbf{G} is pointing vertically downwards, the vector $\boldsymbol{\Omega}_1$ is pointing south and has the magnitude

$$(27) \quad |\boldsymbol{\Omega}_1| = \frac{2}{c^2}|\boldsymbol{\gamma}|G = \frac{2\omega}{c} \cdot \frac{rG}{c^2},$$

$$r = \sqrt{x^2 + y^2}.$$

If the loop is a plane curve, the surface Σ in (21) can be chosen to be a plane, and by (21) and (23) we get

$$(28) \quad \Delta t = \frac{2A}{c}[\boldsymbol{\Omega} \cdot \mathbf{n} + \boldsymbol{\Omega}_1 \cdot \mathbf{n}],$$

where A is the area enclosed by the loop. From (24) and (27) we see that $|\boldsymbol{\Omega}_1|$ is much smaller than $|\boldsymbol{\Omega}|$, by a factor

$$\frac{rG}{c^2} \approx \frac{6 \cdot 10^8 \cdot 10^3}{9 \cdot 10^{20}} \approx 10^{-9}.$$

If we neglect the second term in (28) entirely, we get for the phase difference ΔF of two waves of frequency ν after one turn in a horizontal plane

$$(29) \quad \Delta F = \nu \Delta t = \frac{4\nu\omega A \sin b}{c^2} = \frac{4\omega A}{\lambda c} \sin b,$$

where b is the geographical latitude. For visible light with $\lambda = 4 \cdot 10^{-5}$ cm we get

$$(30) \quad \Delta F \approx 10^{-9} A .$$

With an area of the order of $A = 1\,000$ m², the phase difference will then be of observable magnitude. This first order effect was observed and verified by MICHELSON [7] in 1925 by means of a large interferometer.

However, unfortunately, this experiment cannot be regarded as a real test of the general theory of relativity, since the first order effect depends only on the « non-permanent » part of the gravitational field of the earth originating from the rotation of the earth around its axis. It is true that the principle of equivalence claims that there is no essential difference between such a type of gravitational field and the « permanent » fields due to the mass of the earth; but the fact remains that the formula (27) in first approximation is obtained also in the absolute ether theory, at least for light in vacuo. In fact, we get for the velocity of light in empty space in a system of reference which rotates with the angular velocity ω with respect to the absolute ether system

$$(31) \quad w = c [\sqrt{1 + (\boldsymbol{\gamma}' \cdot \mathbf{e})^2} - \boldsymbol{\gamma}' \cdot \mathbf{e}]$$

with

$$(32) \quad \boldsymbol{\gamma}' = \left(-\frac{\omega y}{c}, \frac{\omega x}{c}, 0 \right).$$

To the first order in $\boldsymbol{\gamma}'$, (31) then leads to the formula (29). In order to obtain a real test of the general theory of relativity, it would be necessary to measure the effect due to the vector $\boldsymbol{\Omega}_1$ in (28), which depends on \mathbf{G} , i.e. on the « permanent » gravitational field as well as on the field originating from the rotation of the earth. This would require a 10^9 times larger area A . Even if it could be arranged so that the signals are making 100 turns instead of just one, the required diameter of the loop would have to be of the order of 100 km. In spite of the marvellous development of radar technique in recent years, it does not therefore seem possible in the foreseeable future to measure any truly relativistic effect of the gravitational field of the earth on the velocity of propagation of radiation.

Thus, it seems that the influence of gravitational fields on the rate of atomic clocks provides the only effects which lend themselves to a check by terrestrial experiments in the near future. On the other hand, a further development of the accuracy of these instruments is probably the only way to eventually checking the non-linear effects of general relativity, which are the really characteristic features of this theory, distinguishing it from the various linear theories of gravitation. However, the accuracy required to open up

this entirely new world is of the order of 10^{-18} , which is the order of magnitude of the non-linear terms in the expression for the rate of standard clocks. If such a high precision of the atomic clocks should ever be obtained, one would have to take into account a number of other effects, the largest of which is the influence of the gravitational field of the moon. In fact, the influence of this field on the quantity Δ in (14) is of the order of 10^{-17} . For two antipodal clocks on the earth, the relative difference in rate due to the gravitational field on the moon is even as large as 10^{-14} . On account of the rotation of the earth around its axis, this effect will obviously give rise to a diurnal variation of the relative rates of the clocks of corresponding magnitude.

APPENDIX A

Emission of photons from an atomic system moving in a gravitational field.

In an arbitrary system of space-time coordinates $(x^i) = (x^t, ct)$ with the four-dimensional interval

$$(A.1) \quad ds^2 = g_{ik} dx^i dx^k,$$

the dynamical action of the gravitational field is determined by the scalar potential χ and the vector potential γ_i defined by

$$(A.2) \quad g_{41} = -(1 + 2\chi/c^2); \quad \gamma_i = g_{i1}/\sqrt{-g_{44}}.$$

(Latin indices are running from 1 to 4, while Greek indices, which denote components of spatial vectors and tensors, are running from 1 to 3, only). Further, the influence of the gravitational field on the spatial geometry in the system of reference considered is described by the spatial metric tensor $\gamma_{\alpha\beta}$ entering in the line element

$$(A.3) \quad d\sigma^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta; \quad \gamma_{\alpha\beta} = g_{\alpha\beta} + \gamma_\alpha \gamma_\beta.$$

Hence,

$$(A.4) \quad ds^2 = d\sigma^2 - (c' dt - \gamma_i dx^i)^2$$

with c' given by Eq. (5) in the text.

In the following we shall only have to consider stationary fields where χ , γ and $\gamma_{\alpha\beta}$ are time-independent. A particle, elementary or composite, which in a rest system of inertia has a mass \tilde{m}_0 will, when moving with the velocity \mathbf{u} in a gravitational field, have a mass m , momentum vector \mathbf{p} , and energy E

given by the following equations [8]:

$$(A.5) \quad \begin{cases} m = \dot{m}_0 / [(\sqrt{1 + 2\chi/c^2} - \boldsymbol{\gamma} \cdot \mathbf{u}/c)^2 - u^2/c^2]^{\frac{1}{2}}, \\ \mathbf{p} = m\mathbf{u}, \\ E = mc'(c' - \boldsymbol{\gamma} \cdot \mathbf{u}). \end{cases}$$

Here, \mathbf{u} is the velocity vector with contravariant components $u^i = dx^i/dt$, $\boldsymbol{\gamma} \cdot \mathbf{u} = \gamma_i u^i$ is the scalar product of the vector potential $\boldsymbol{\gamma}$ and the velocity vector, and $u^2 = \gamma_{\alpha\beta} u^\alpha u^\beta$ is the square of the velocity. In a stationary gravitational field, the energy E is a constant of the motion for a freely falling particle. A simple calculation shows that E and $|\mathbf{p}|^2 = p_i p^i = m^2 u^2$ are connected by a similar equation as energy and momentum of a free particle in a system of inertia, viz.

$$(A.6) \quad |\mathbf{p}|^2 - (E/c')^2 = -\dot{m}_0^2 c^2.$$

This follows at once from (A.5), noticing that m may be written

$$(A.7) \quad m = \dot{m}_0 c' / \sqrt{(c' - \boldsymbol{\gamma} \cdot \mathbf{u})^2 - u^2/c^2}.$$

The velocity of light w in the gravitational field (A.1)-(A.4) depends on the direction of propagation e and is equal to [9]

$$(A.8) \quad w = c' / (1 + \boldsymbol{\gamma} \cdot \mathbf{e}),$$

i.e.

$$(A.9) \quad w = c' - \boldsymbol{\gamma} \cdot \mathbf{w}; \quad \mathbf{w} = w\mathbf{e}.$$

If we let \mathbf{u} in (A.7) approach the velocity of light \mathbf{w} , the denominator goes to zero and we have to let \dot{m}_0 go to zero also in order to get a finite value for the mass m . In this way we get from (A.5) and (A.9) for the momentum and energy of a photon in a gravitational field

$$(A.10) \quad \begin{cases} \mathbf{p} = m\mathbf{w} = m w \mathbf{e}, \\ E = mc'w. \end{cases}$$

Hence,

$$p = |\mathbf{p}| = E/c',$$

in accordance with (A.6) in the limit $m_0 \rightarrow 0$.

If ν is the frequency of the radiation to which the photon belongs we have for its energy and momentum

$$(A.12) \quad E = h\nu; \quad \mathbf{p} = \frac{h\nu}{c'} \cdot \mathbf{e},$$

where h is Planck's constant.

A photon travelling through a gravitational field may thus be treated as

a «freely falling» particle of rest mass zero. If the field is stationary its energy, and thus its frequency, is constant along the path. The red shift of spectral lines emitted at the surface of the sun is therefore, according to this theory, *not* due to a change of frequency of the light on its way to the earth (as in some of the popular explanations of this phenomenon), but rather to a difference in the energy levels of an atom when it is placed at different gravitational potentials.

Let us now consider a molecule in an excited state moving through a gravitational field with the velocity \mathbf{u} . When the molecule makes a transition to the ground state a photon is emitted, a process in which the total energy and momentum are conserved.

Let $\overset{\circ}{m}_0$, \mathbf{p} and E be the proper mass, momentum and energy, respectively, of the molecule before the process. They are connected by the equations (A.5) and (A.6). If \bar{m}_0 , $\bar{\mathbf{p}}$, \bar{E} are the corresponding quantities after the emission of the photon, we have

$$(A.13) \quad \mathbf{p} - \bar{\mathbf{p}} = \frac{h\nu}{c'} \mathbf{e}; \quad E - \bar{E} = h\nu.$$

Elimination of \mathbf{p} and E from the equations (A.13), (A.6), and the analogous equation

$$(A.14) \quad |\bar{\mathbf{p}}|^2 - (\bar{E}/c')^2 = -\bar{m}_0^2 c^2$$

gives after a simple calculation

$$(A.15) \quad \nu = \frac{(\overset{\circ}{m}_0^2 - \bar{m}_0^2)c^2 c'^2}{2hE[1 - c'(\mathbf{p} \cdot \mathbf{e})/E]}.$$

If the initial velocity \mathbf{u} of the molecule is zero the frequency emitted is, according to (A.5) and (A.15),

$$(A.16) \quad \nu_0 = \frac{(m_0^2 - \bar{m}_0^2)c^2 c'^2}{2hE_0},$$

where

$$(A.17) \quad E_0 = m_0 c'^2 = m_0 c^2 \sqrt{1 + 2\chi/c^2}$$

is the rest energy of the molecule at a place of gravitational potential χ . From (A.15) and (A.16) we get the formula (4), i.e.

$$(A.18) \quad \nu = \nu_0 \frac{E_0}{E[1 - c'(\mathbf{p} \cdot \mathbf{e})/E]}.$$

Further we get from (A.16), (A.17) and (5)

$$(A.19) \quad \nu_0 = \overset{\circ}{\nu}_0 \sqrt{1 + 2\chi/c^2},$$

where

$$(A.20) \quad \dot{\nu}_0 = \frac{(\overset{\circ}{m}_0^2 - \bar{m}_0^2)c^2}{2h\bar{m}_0}$$

is the frequency emitted when also χ is equal to zero, i.e. when the molecule is at rest in a system of inertia. (A.19) is identical with the equation (6). Expansion of E and \mathbf{p} in powers of the velocity \mathbf{u} leads at once from (A.18) to the expansion (7) used in the previous discussion.

APPENDIX B

Satellite problems.

Since the earth is falling freely in the gravitational field of the sun it is clear that we can introduce a system S of space-time coordinates (x, y, z, t) which at all times is a local system of inertia for the centre of gravity of the system earth-moon with the spatial coordinates $x = y = z = 0$. In the linear approximation, we get for the interval in the space surrounding the earth

$$(B.1) \quad ds^2 = [1 - 2(\chi_1 + \chi_{\text{moon}})/c^2 + O(10^{-12})](dx^2 + dy^2 + dz^2) - [1 + 2(\chi_1 + \chi_{\text{moon}})/c^2 + O(10^{-16})]c^2 dt^2,$$

where

$$(B.2) \quad \chi_1(x, y, z) = -k \int \frac{\mu(x', y', z') dx' dy' dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

is the permanent scalar gravitational potential due to the mass distribution μ of the earth and χ_{moon} is the corresponding potential of the moon. Thus neglecting terms of the order 10^{-16} in g_{44} and terms of the order 10^{-12} in the remaining components of the metric tensor g_{ik} , the influence of the gravitational field of the sun has disappeared completely in S . This is a good approximation for distances from the centre of the earth of at least 10 times the radius of the earth, as is readily seen by actually performing the transformation from the system in which the sun is at rest to the system S which follows the motion of the earth.

For a spherical earth we would get

$$(B.3) \quad \chi_1 = -\frac{kM}{r}, \quad r = \sqrt{x^2 + y^2 + z^2},$$

and this approximation is sufficient for our purposes. At the surface of the

earth the order of magnitude of the potentials in (B.1) is

$$(B.4) \quad \chi_1/c^2 \approx 10^{-9}, \quad \chi_{\text{moon}}/c^2 \approx 10^{-14}.$$

As a first approximation we may neglect the weak field of the moon, and the gravitational field in S is then time-independent.

Now consider a standard clock placed on an artificial satellite circling around the earth.

Since the vector potential is zero in S , we get for the proper time τ of the clock [10], on account of (A.5),

$$(B.5) \quad d\tau = dt \sqrt{1 + 2\chi_1/c^2 - u^2/c^2} = dt \frac{\dot{m}_0}{m} = dt \frac{\dot{m}_0 c^2}{E} (1 + 2\chi_1/c^2),$$

where m_0 and E are the proper mass and the energy, respectively, of the satellite in its orbit. The factor $\dot{m}_0 c^2/E$ is a constant of the motion, and by means of the equations of motion the exact formula (B.5) allows us to calculate the increase $\Delta\tau$ in the reading of the clock for the time interval Δt between two positions of closest approach of the satellite and the earth. However, since the effect in which we are interested is small, we can replace the exact equations of motion by the corresponding approximate equations of Newtonian mechanics. If we introduce polar coordinates (r, φ) in the plane of the orbit, we have in this approximation

$$(B.6) \quad \frac{1}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) - \frac{kM}{r} = \frac{E}{m_0} - c^2 \equiv -\frac{B}{2},$$

$$(B.7) \quad r^2 \dot{\varphi} = C,$$

where B and C are constants of integration.

The first equation is obtained from (A.5) by expanding the energy in powers of χ_1/c^2 and u^2/c^2 and retaining only the terms linear in these quantities. The second equation is the usual angular momentum integral which is a consequence of the spherical symmetry of our problem. The equations (B.6), (B.7) differ from the corresponding exact equations only by terms which are 10^{-9} times smaller than the terms retained. By elimination of $\dot{\varphi}$ we get the radial equation of the Kepler motion

$$(B.8) \quad \dot{r} = \pm \frac{\sqrt{B}}{r} \sqrt{-r^2 + \frac{2kMr}{B} - \frac{C^2}{B}} = \pm \frac{\sqrt{B}}{r} \sqrt{(r-r_1)(r_2-r)},$$

where r_1 and r_2 , the radii of smallest and largest distances in the orbit, are connected with the constants of integration B and C by

$$(B.9) \quad r_1 + r_2 = \frac{2kM}{B}, \quad r_1 r_2 = \frac{C^2}{B}.$$

Hence

$$(B.10) \quad \frac{E}{\dot{m}_0 c^2} \equiv 1 - \frac{B}{2c^2} = 1 - \frac{kM}{(r_1 + r_2)c^2}.$$

For the time interval Δt between two positions of closest approach we get from (B.8)

$$(B.11) \quad \Delta t = \frac{2}{\sqrt{B}} \int_{r_1}^{r_2} \frac{r dr}{\sqrt{(r - r_1)(r_2 - r)}} = \frac{(r_1 + r_2)\pi}{\sqrt{B}} = \frac{\pi(r_1 + r_2)^{\frac{3}{2}}}{\sqrt{2kM}} [1 + O(10^{-9})],$$

where, in the last expression, the order of magnitude of the different terms neglected in the calculation is indicated by $O(10^{-9})$. In the same approximation we get from (B.5), (B.8), and (B.10) for the increase in the proper time of the satellite clock

$$(B.12) \quad \begin{aligned} \Delta\tau_{\text{sat.}} &= \frac{m_0 c^2}{E} \int_0^{\Delta t} \left(1 - \frac{2kM}{c^2 r}\right) dt = \\ &= \left(1 + \frac{kM}{(r_1 + r_2)c^2}\right) \Delta t - \frac{4kM}{c^2 \sqrt{B}} \int_{r_1}^{r_2} \frac{dr [1 + O(10^{-9})]}{\sqrt{(r - r_1)(r_2 - r)}}. \end{aligned}$$

Since the integral in the last term of (B.12) has the value π , this term may, by (B.10), (B.11), be written

$$(B.13) \quad - \frac{4kM(r_1 + r_2)^{\frac{3}{2}}\pi}{c^2 \sqrt{2kM}} [1 + O(10^{-9})] = - \frac{4kM\Delta t}{(r_1 + r_2)c^2} [1 + O(10^{-9})].$$

Hence,

$$(B.14) \quad \frac{\Delta\tau_{\text{sat.}} - \Delta t}{\Delta t} = - \frac{3kM}{(r_1 + r_2)c^2} [1 + O(10^{-9})].$$

A standard clock at rest on the surface of the earth has a velocity $r'\omega$ relative to S , where r' is the distance to the axis of rotation of the earth, and ω is the angular velocity in this rotation.

Hence

$$\frac{u^2}{c^2} \approx \left(\frac{r'\omega}{c}\right)^2 \approx 10^{-12} \approx \frac{kM}{c^2 r} \cdot 10^{-3}.$$

From the general formula (B.5) we therefore get for the rate of the earth clock

$$(B.15) \quad \frac{\Delta\tau_{\text{earth}} - \Delta t}{\Delta t} = \frac{\chi_1 - \frac{1}{2}r'^2\omega^2}{c^2} = \frac{\chi}{c^2} = - \frac{kM}{c^2 r} [1 + O(10^{-3})],$$

where $\chi = \chi_1 - \frac{1}{2}r'^2\omega^2$ is the scalar potential at the surface of the earth in

the earth-fixed system S_E introduced in Appendix C. From (B.14), (B.15) we get the formula

$$(B.16) \quad \frac{\Delta\tau_{\text{sat.}} - \Delta\tau_{\text{earth}}}{\Delta\tau_{\text{earth}}} = \frac{kM}{c^2} \left(\frac{1}{r} - \frac{3}{r_1 + r_2} \right),$$

i.e. the equation (15).

APPENDIX C

Propagation of radiation through a medium.

We shall now consider the propagation of radiation through a medium at rest in a system of reference S in which the gravitational field (A.1), (A.4) is stationary. If \dot{w} is the velocity of propagation of the radiation with respect to a system of inertia in which the medium is at rest, we have for the time-track of the signal

$$(C.1) \quad ds^2 = g_{in} dx^i dx^k = d\dot{X}^i d\dot{X}^i - c^2 d\dot{T}^2,$$

where $\dot{X}^i = (\dot{X}^i, c\dot{T})$ are the space-time coordinates in a local rest system of inertia \dot{S} for the space-time point in question. Since $\dot{w}^2 = \sum_{i=1}^3 (d\dot{X}^i/d\dot{T})^2$ and $w^i = dx^i/dt$ are the contravariant components of the velocity of the signal in the gravitational field, (C.1) may be written, by means of (A.1)–(A.4),

$$(C.2) \quad w^2 - (c' - \gamma_i w^i)^2 = -c^2(1 - \dot{w}^2/c^2) \left(\frac{d\dot{T}}{dt} \right)^2,$$

where $w = d\sigma/dt = \sqrt{\gamma_{ik} w^i w^k}$ is the magnitude of the velocity of the signal in the gravitational field. Since the systems of reference S and \dot{S} are at rest with respect to each other at the space-time point P considered, the transformation connecting the space-time coordinates in S and \dot{S} must be such that

$$(C.3) \quad c d\dot{T} = -\gamma_i(P) dx^i + c \sqrt{1 + 2\chi(P)/c^2} dt = dt \left(c' - \gamma_i \frac{dx^i}{dt} \right),$$

where γ_i and χ in c' are the values of the potentials at the space-time point P . Combining (C.2) and (C.3) we get

$$w^2 - (c' - \gamma_i w^i)^2 = -(1 - \dot{w}^2/c^2)(c' - \gamma_i w^i)^2,$$

or

$$(C.4) \quad w = \frac{\dot{w}}{c} (c' - \gamma_i w^i).$$

Putting $w^i = w e^i$, where e^i are the contravariant components of a unit vector in the direction of propagation of the radiation, and solving with respect to w gives

$$(C.5) \quad w = \frac{c' w / c}{1 + (\dot{w}/c)(\boldsymbol{\gamma} \cdot \mathbf{e})} = \frac{c'}{(c/\dot{w}) + \boldsymbol{\gamma} \cdot \mathbf{e}},$$

which is the equation (17) of the text.

In an earth-fixed system $S_E: (x', y', z', t')$, connected with the coordinates of the system $S: (x, y, z, t)$ used in Appendix B by the transformation

$$(C.6) \quad \begin{cases} x = x' \cos \omega t' - y' \sin \omega t', \\ y = x' \sin \omega t' + y' \cos \omega t', \\ z' = z, \\ t' = t, \end{cases}$$

we have, disregarding the influence of the moon,

$$(C.7) \quad ds^2 = (1 - 2\chi_1/c^2)(dx'^2 + dy'^2 + dz'^2) - 2\omega y' dx' dt' + 2\omega x' dy' dt' - (1 + 2\chi_1/c^2 - r'^2\omega^2/c^2)c^2 dt'^2.$$

Here, ω is the angular velocity of the earth in its rotation around the z -axis. Thus, by (A.2), we get for the potentials in S_E

$$(C.8) \quad \chi = \chi_1 - \frac{1}{2}r'^2\omega^2; \quad \gamma_i = \left(-\frac{\omega y'}{c'}, \frac{\omega x'}{c'}, 0 \right),$$

which is in accordance with (22) after dropping the primes on the coordinates in S_E . Further, the spatial metric tensor may be written

$$(C.9) \quad \gamma_{ik} = (1 - 2\chi_1/c^2)\delta_{ik} + \gamma_i \gamma_k = (1 - 2\chi/c^2)\delta_{ik},$$

if we consistently neglect small terms of the order 10^{-12} . For the determinant $\gamma \equiv |\gamma_{ik}|$ we therefore get

$$(C.10) \quad \gamma = (1 - 2\chi/c^2)^3.$$

Using the general formulae defining the components of curl \mathbf{a} and of the

vector product $\mathbf{a} \times \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} [11]

$$(C.11) \quad \text{curl } \mathbf{a} = \frac{1}{\sqrt{\gamma}} \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}, \frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1}, \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right),$$

$$(C.12) \quad (\mathbf{a} \times \mathbf{b})^i = \frac{1}{\sqrt{\gamma}} (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1),$$

it is easily seen that the curl of the vector (20) is of the form (23) with

$$(C.13) \quad \Omega^i = \frac{1}{(1 - 2\chi/c^2)^{\frac{3}{2}}(1 + 2\chi/c^2)} \left(0, 0, \frac{2\omega}{c} \right),$$

and Ω_1 given by (25). In the expression for Ω_1 we have consistently neglected a factor of the order $[1 + O(10^{-9})]$. For the norm of the vector Ω we get

$$(C.14) \quad |\Omega|^2 = \gamma_{\kappa} \Omega^{\kappa} \Omega^{\kappa} = \\ = (1 - 2\chi/c^2) \frac{4\omega^2/c^2}{(1 - 2\chi/c^2)^3(1 + 2\chi/c^2)^2} = \frac{4\omega^2}{c^2} [1 + O(10^{-18})],$$

in accordance with the equation (24) in the text.

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