tributed in some places than in others, of infinitely small average density through the whole of infinite space. In regions where the density was then greater than in neighbouring regions, the density would become greater still; in places of less density, the density would become less; and large regions would quickly become void or nearly void of atoms. These large void regions would extend so as to completely surround regions of greater density. In some part or parts of each cluster of atoms thus isolated, condensation would go on by motions in all directions not generally convergent to points, and with no perceptible mutual influence between the atoms until the density becomes something like 10^{-6} of our ordinary atmospheric density, when mutual influence by collisions would begin to become practically effective. Each collision would give rise to a train of waves in ether. These waves would carry away energy, spreading it out through the void ether of infinite space. The loss of energy, thus taken away from the atoms, would reduce large condensing clusters to the condition of gas in equilibrium * under the influence of its own gravity only, or rotating like our sun or moving at moderate speeds as in spiral nebulas, &c. Gravitational condensation would at first produce rise of temperature, followed later by cooling and ultimately freezing, giving solid bodies; collisions between which will produce meteoric stones such as we see them. We cannot regard as probable that these lumps of broken-looking solid matter (something like the broken stones used on our macadamised roads) are primitive forms in which matter was created. Hence we are forced, in this twentieth century, to views regarding the atomic origin of all things closely resembling those presented by Democritus, Epicurus, and their majestic Roman poetic expositor, Lucretius.

II. *On the Michelson-Morley Experiment relating to the Drift of the Æther.* By W. M. Hicks, *F.R.S.*†

[Plate I.]

IN the following pages it is proposed to consider in detail the general theory of the experiment by which Messrs. Michelson & Morley ‡ attempted to decide the question of the rest or motion of the æther when a material body moves through it. The theory is not so simple as it may appear at

* Homer Lane, American Journal of Science, 1870, p. 57; Sir W Thomson, Phil. Mag. March 1887, p. 287.

† Communicated by the Author.

‡ Phil. Mag. Dec. 1887.

first sight owing to the changes produced by actual reflexion at a moving surface. The correction due to alteration in the angle of reflexion was first introduced by Lorentz, and was taken account of in the joint paper by Michelson & Morley in 1887. But reflexion produces also a change in the wave-length of the reflected light. Further, when the source of light moves with the apparatus, the light incident at any instant on a plate does not come from the position occupied by the source at that instant, but from a point which it occupied at some interval before; consequently the angle of incidence alters by a small quantity of the first order as the direction of drift of the apparatus changes.

The present investigation is undertaken with the view of making these corrections. As in a theory of this kind it does not seem legitimate to assume beforehand that the ratio of the velocity of the earth through the æther to that of light is extremely small, the general theory is worked out without approximation. It is interesting to see how comparatively simple this complete theory is. The principal result of the correction is that in the experiment of Michelson & Morley the effect to be expected is the reverse of that hitherto supposed.

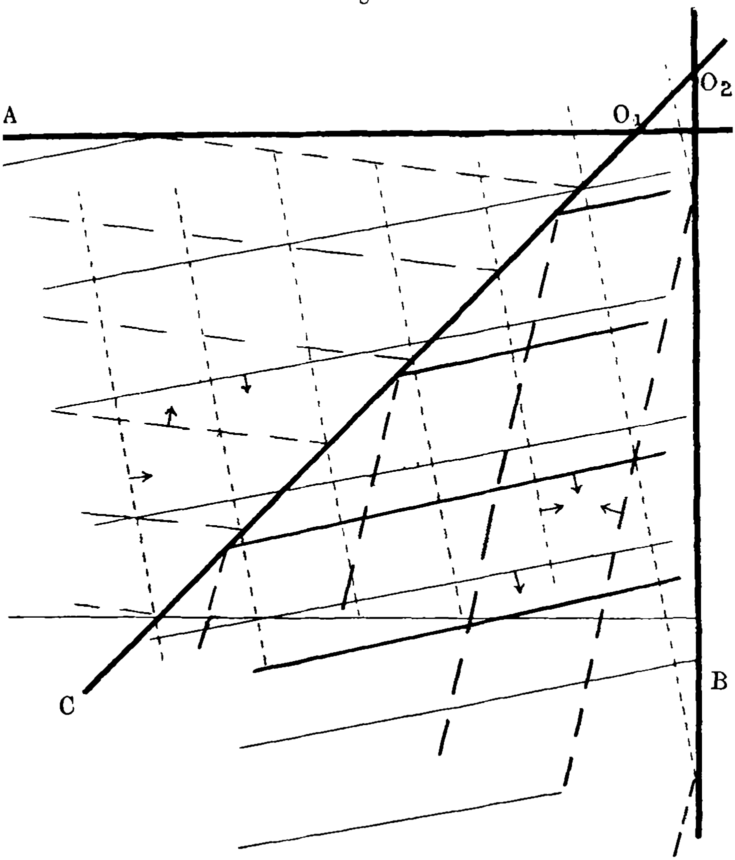
In the actual experiment the light from a flame in the focal plane of a lens falls on a small plane-glass plate by which it is divided, and the two portions are afterwards reflected back to the plate by two small plane mirrors. We shall suppose in the first place that the source is a point of light or a vertical slit; so that the incident light consists of a single train of parallel waves.

We have thus two mirrors inclined at any given angle, and between them a transparent semi-reflecting plate, whose plane does not in general pass through the intersection of the mirrors. In practice we utilize only small portions of these planes, but in considering the general theory it is best to deal with the complete system of planes and to consider the phenomena taking place in the angular space between them.

1. Consider the state of the æther at any given instant, and draw the wave diagram, that is a diagram of lines representing all the crests (or every nth crest).

In fig. 1, A, B represent the planes of the mirrors, C that of the plate. On the plate is incident a train of waves (dotted lines), which produces at the plate a train of reflected waves (thin broken lines) and of transmitted waves (dotted lines). The first set are incident on the mirror plane A and produce a train of reflected waves (thin continuous lines). The transmitted set fall on the mirror plane B, and produce a reflected train (thick broken lines), these, after reflexion from the glass plate, produce another train (thick continuous lines). The two sets of waves, represented by the thin and thick lines, interfere and produce the fringes whose laws we

Fig. 1.



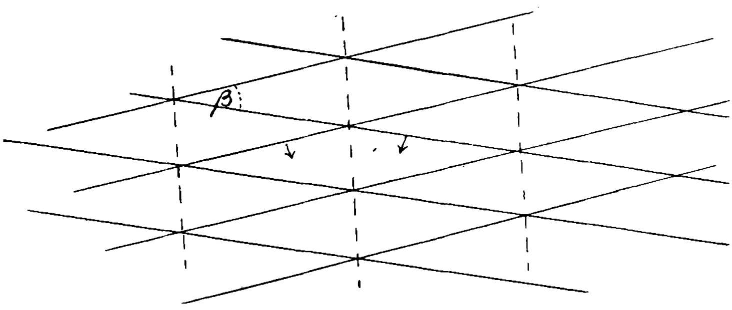
have to investigate. The figure gives an instantaneous picture of the configuration of the waves in the æther for a special case of motion and disposition of the planes*. The diagram is complicated by the number of lines. But the different trains can be followed easily, and afterwards the eye fixed on the continuous lines only. The essential point

* The figure is drawn to scale for the case of $U=\frac{1}{5}V$, drift makes angle of 53° 8′ ($\tan a=\frac{4}{3}$) with OA. $CO_1A=44°\ 30'$, mirrors at 90° to one another.

to be noticed about this wave-diagram is that precisely the same diagram will serve for every instant of the motion; only in this case the lines do not represent crests, but all belong to the same phase of the respective waves, this common phase differing with the instant for which the diagram is drawn. This is because the change produced by reflexion so modifies the reflected waves (as will be seen shortly) that the incident crests and their corresponding reflected crests always intersect along the surface as it moves through the æther. If the diagram were drawn with angles of reflexion and incidence equal at one instant, at the next they would be thrown into confusion. This permanence of type is the basis of the development of the theory adopted in the following general reasoning. The diagram gives an instantaneous view. If an observer is fixed in the æther he sees that in this picture the waves have different wave-lengths, but that they advance with the same velocity V, and the apparatus moves with U. If the observer is fixed to the apparatus he sees that in this picture the apparatus is fixed but that each system of waves advances with a different velocity, yet in all cases such, that as they reach him their frequencies are the same—the longer waves have the greater apparent velocity of propagation.

2. It will be best to consider first the case of no drift of the æther. Here the æther between B and C will be mapped out by a network of lines representing crests of two sets of

Fig. 2.



waves, the meshes of which, in case of no motion, are equal-sided parallelograms. The diagonals which lie between the advancing sides of the waves are points of maximum disturbance. They form a system of parallel straight lines bisecting the angles between the wave-fronts, and at a distance from one another $c=\lambda/\left(2\sin\frac{\beta}{2}\right)$, where λ is the common wave-length and β the angle between the two sets of wave-fronts. For convenience we shall call these lines maximal lines. If a screen be placed perpendicularly to them we shall get a fringe, the intersections with the maximals being the centres of the bright bands, and the distance between the bands being c. If the screen is placed at an angle γ to these lines, the breadth of the bands is $c/\sin\gamma$.

All the waves interfering at points along a given maximal differ in phase by the same integral number of wave-lengths. We can therefore designate any maximal by this number. It is important to be able to determine it.

If the plate and the first mirror intersect at O_1 (fig. 1), the phase of the reflected wave of the first train at O_1 is the same as that of the incident. Let p_1 denote the perpendicular distance of any point P from the wave-front through O_1. Then the phase of the first train at $P=p_1/\lambda+$ phase of incident light at O_1. Similarly if the plate and the second mirror intersect at O_2, and p_2 denote the distance of P from the wave-front of the second train through O_2, the phase of the second train at $P=p_2/\lambda+$phase of incident light at O_2. If then the distance $O_1\,O_2$ resolved perpendicularly to the incident system be q, the interfering waves at P differ in phase by $(p_2+q-p_1)/\lambda$. This is the amount by which the second train is ahead of the first. If this be zero we have the central maximal, which is the same for all colours. Consequently, when white light is used it is only near this maximal that fringes will be visible.

It follows from this that if the planes of the plate and the mirrors intersect in a point the central maximal will pass through that point. There is no need to consider specially the quantitative theory for no drift, as it will be included in the more general case of motion to follow.

3. Taking now the case where the apparatus moves in an æther at rest, the wave-diagram gives as before an instantaneous state of the æther. But now the conditions are very different. In the first place, as will be shown later, when the two reflected systems are inclined their wave-lengths will in general be different, whilst if the wave-lengths are equal the two systems must be parallel. Thus to a person fixed in the æther, no fringes can *in general* be seen, either because the waves which ought to interfere have different frequencies, or because when they have the same, the interference produces a uniform change of illumination and not bands of finite breadth. This is not the case, however, to a person who

moves with the apparatus. For consider a point P fixed relatively to it. At every instant the phases of the two wave-fronts at P (each phase measured as a fraction of its corresponding period) differ by the same amount. Consequently as P moves through the æther the two waves at P interfere. The *apparent* frequency to an observer at P of each system must consequently be the same. A formal proof of this is given below.

So far then as the phenomena at points moving with the apparatus are considered, we may consider the space as divided up into a network of parallelograms (not now equal-sided) with a corresponding set of maximal lines. These maximal lines are not fixed in the æther, but move as if fixed to the apparatus. The intersections of these maximals with a screen moving with the apparatus give the fringes, as in the case of no motion. The maximal lines will not now be equally inclined to the two wave-fronts. If as before e denote the distance between the maximals, it can easily be shown that

$$e=\frac{\lambda_1\lambda_2}{\sqrt{\{(\lambda_1-\lambda_2)^2+4\lambda_1\lambda_2\sin^2\frac{1}{2}\beta\}}}.$$

The maximal lines move as a rigid system attached to the apparatus. They give the phenomena taking place in the fixed æther as well. If a screen be fixed in the æther the intersections with the maximals move, and we get fringes which drift along it, and which therefore in general produce the effect of white light on the eye owing to their rapid motion. If, however, the direction of drift be parallel to a maximal line, there is no drift of fringes on the screen, and the eye will see a fixed fringe. If the drift slightly alters, the fringe will begin to move slowly along the screen. The observation naturally cannot be made, but the result is important, because, if the fringes exist, the interfering waves must have equal frequencies. Hence the two sets of wave-lengths will be equal when the drift is parallel to a maximal. That this is actually the case the formula to be developed later will show.

4. If a photographic plate be exposed in any position, an image of the fringe there situated will be impressed. If, however, the eye be focussed on the same plane, either directly or by any optical instrument, the fringe observed will not be the same as that impressed on the plate. The rays through any point of the object fringe all pass through the optical image on the retina, and all traverse the same optical distance,

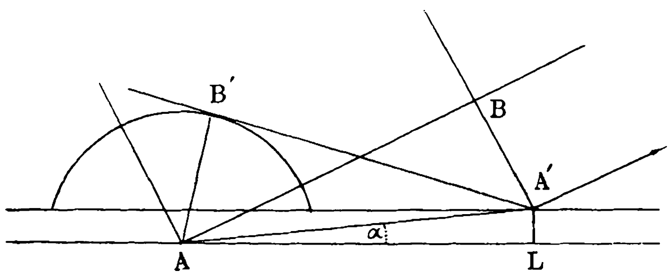
say D′. At the object the two interfering waves have a certain phase-difference, but in space their wave-lengths are different, and in travelling over the same distance D′ their phase-differences are changed by the amount $D'\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)$. Consequently the optical image of a bright line is not itself a bright line ; or, conversely, the bright central band in the image is the optical image of a point on the original fringe to one side of its central band. This result is important, as in certain cases it may, as will be shown later, entirely modify the nature of the changes produced as the direction of drift of the apparatus alters. Again, superposed on this there will be an aberration effect.

So much for the general qualitative theory of the phenomenon. It remains now to obtain the quantitative laws.

Reflexion of a Plane Wave at a Moving Mirror.

5. *Angle of Reflexion.*—In the figure let AB denote a wave-front incident at A on the mirror AL. Let V denote the

Fig. 3.



velocity of light, v that of the mirror perpendicular to itself. When the mirror has advanced $A'L=vt$ suppose AB to have advanced so as to be incident at A′. During the interval t, *provided the æther be at rest*, the disturbance produced at A will have spread to the spherical surface B′, which lies outside the new position of the mirror. What happens between the old and new positions of the mirror does not concern us. Hence, with the usual reasoning, the reflected wave-front through A′ is A′B′ (the fig. explains itself). Let ϕ, ϕ' be the angles of incidence and of reflexion, and let α denote the angle A′AL.

Then $$\mathrm{BAA'}=\mathrm{B'A'A},$$

i. e. $$\phi-\alpha=\phi'+\alpha,$$

or $$\alpha=\frac{\phi-\phi'}{2},$$

$$\mathrm{BAA'}=\phi-\alpha=\frac{\phi+\phi'}{2}.$$

Again, $$\frac{\sin \mathrm{BAA'}}{\sin \alpha}=\frac{\mathrm{A'B}}{\mathrm{A'L}}=\frac{\mathrm{V}}{v}$$

or $$\frac{\sin\frac{\phi+\phi'}{2}}{\sin\frac{\phi-\phi'}{2}}=\frac{\mathrm{V}}{v},$$

whence $$\tan\tfrac{1}{2}\phi'=\frac{\mathrm{V}-v}{\mathrm{V}+v}\tan\tfrac{1}{2}\phi, \quad . \quad . \quad . \quad . \quad (1)$$

which gives the law of reflexion. In this it is to be remembered that v is to be regarded as positive when the plane moves towards the incident light.

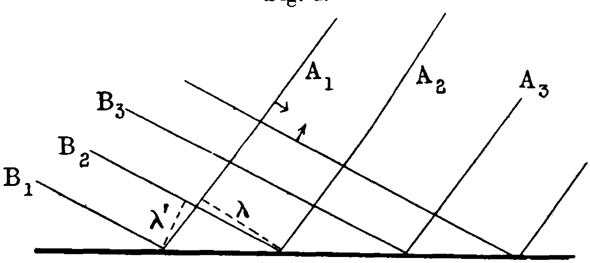
It can easily be shown from the above that if D_v denotes $\surd\{\mathrm{V}^2+v^2+2\mathrm{V}v\cos\phi\}$,

$$e^{\frac{\phi'\iota}{2}}=\frac{\mathrm{V}e^{\frac{\phi\iota}{2}}+ve^{-\frac{\phi\iota}{2}}}{\mathrm{D}_v},$$

a formula which later will be found very useful.

Change in Wave-length.—In fig. 4 let A_1 ... represent a train of wave-crests incident on the mirror at any moment.

Fig. 4.



B_1... the corresponding reflected wave-crests. The figure shows at once that, λ', λ denoting the wave-lengths of the reflected and incident waves respectively,

$$\frac{\lambda'}{\lambda}=\frac{\sin\phi'}{\sin\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

This result is true whatever be the law of reflexion.

Applying the actual law of reflexion above, it follows that

$$\frac{\lambda'}{\lambda}=\frac{V^2-v^2}{V^2+v^2+2Vv\cos\phi}. \quad . \quad . \quad . \quad . \qquad (3)$$

Modification produced on the Light at any point by motion of the source.—Owing to motion of the source the light reaching a point P will not come from the instantaneous position of the source but from the position occupied by it at some time previously. Consequently if S denote the source, P this position, and v the velocity of S perpendicular to SP, the light at P makes an angle θ with SP where

$$V\sin\theta=v.$$

Again, if λ denote the wave-length at P (or V/λ the frequency), and if Λ denote the wave-length if the source is at rest, and u denote the velocity of the source towards P,

$$\lambda=\frac{V\cos\theta-u}{V}\Lambda\cos\theta. \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

6. *Specification of the Configuration.*—We shall call the mirror which receives the light after reflexion at the plate, the first mirror; that which receives it after transmission through the plate the second mirror, and we shall refer all directions to the line through the source parallel to the first mirror.

The normal to the plate makes an angle ϕ (fig. 5) with this datum line, and the normal to the second plate an angle χ (positive as in fig.). Let θ be the inclination of the incident light to this datum line. The angles of incidence on the plate and the second mirror are $\phi+\theta$ and $\chi+\theta$.

Let the angles of reflexion be ϕ' and χ'. Let the angles of incidence and reflexion on the first mirror be ϕ_1, ϕ_1', and on the plate after reflexion at the second mirror be ϕ_2 and ϕ_2'. Also let A, B denote the angles the final wave-fronts make with the first mirror (or the datum line). This specifies the light. Let the apparatus drift with velocity U in a direction making α with the datum, its velocities along and perpendicular thereto being denoted by u, v.

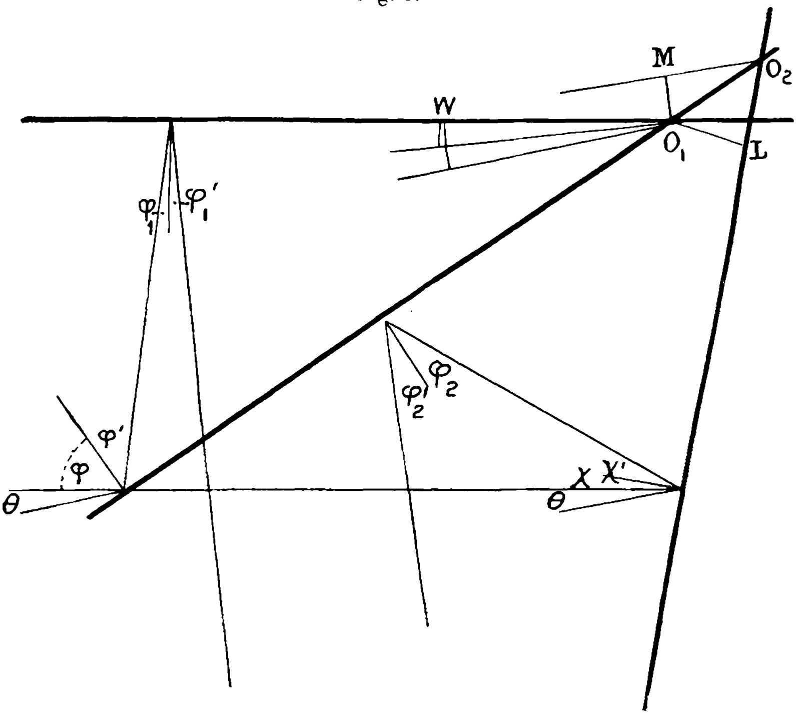
The velocity of the plate perpendicular to itself, and towards the side on which is the source of light, is

$$w = v \sin \phi - u \cos \phi = -\mathrm{U} \cos (\phi + \alpha). \quad . \qquad (5)$$

Similarly that of the second mirror is

$$w' = v \sin \chi - u \cos \chi = -\mathrm{U} \cos (\chi + \alpha). \quad . \qquad (6)$$

Fig. 5.



7. *The Wave-lengths.*—With the above notations, the final wave-lengths will be by (2),

$$\lambda_1 = \frac{\sin \phi'}{\sin (\phi + \theta)} \cdot \frac{\sin \phi_1'}{\sin \phi_1} \lambda,$$

$$\lambda_2 = \frac{\sin \chi'}{\sin (\chi + \theta)} \cdot \frac{\sin \phi_2'}{\sin \phi_2} \lambda,$$

where λ is the wave-length incident on the plate.

Whence by (3)

$$\left.\begin{aligned}\frac{\lambda_1}{\lambda}&=\frac{(V^2-w^2)(V^2-v^2)}{\{V^2+w^2+2wV\cos(\phi+\theta)\}\{V^2+v^2-2vV\cos\phi_1\}}\\ \frac{\lambda_2}{\lambda}&=\frac{(V^2-w^2)(V^2-w'^2)}{\{V^2+w^2-2wV\cos\phi_2\}\{V^2+w'^2+2w'V\cos(\chi+\theta)\}}\end{aligned}\right\}(7)$$

It will be convenient to denote these denominators by D_1^2 and D_2^2.

Now $$\phi_1=\phi+\phi'-\tfrac{1}{2}\pi.$$

Hence

$$D_1^2=(V^2+v^2)\{V^2+w^2+2wV\cos(\phi+\theta)\}$$
$$-2Vv\sin(\phi+\phi')\{V^2+w^2+2wV\cos(\phi+\theta)\}.$$

ϕ' is the reflected angle for incident angle $(\phi+\theta)$ and for vel. $=w$. Therefore

$$\{V^2+w^2+2wV\cos(\phi+\theta)\}\begin{cases}\sin\phi'=(V^2-w^2)\sin(\phi+\theta),\\ \cos\phi'=(V^2+w^2)\cos(\phi+\theta)+2Vw.\end{cases}$$

Substituting it can be easily shown, remembering the values of w, w' given by (5, 6), that

$$D_1^2=V^4-2V^3\cos\phi\{u\cos(\phi+\theta)+v\sin(\phi+\theta)\}$$
$$+V^2(v^2+w^2-4vw\sin\phi)+2vwV\cos\phi(v\cos\theta-u\sin\theta)+v^2w^2.$$

Similarly it may be shown that

$$D_2^2=V^4+2V^3\sin(\phi-\chi)\{u\sin(\theta+\chi-\phi)-v\cos(\theta+\chi-\phi)\}$$
$$+V^2\{w^2+w'^2-4ww'\cos(\phi-\chi)\}$$
$$-2ww'V\sin(\phi-\chi)(u\sin\theta-v\cos\theta)+w^2w'^2.$$

The difference of wave-lengths is given by $d\lambda=\lambda_1-\lambda_2$, where

$$\frac{d\lambda}{\lambda}=(V^2-w^2)\frac{(V^2-v^2)D_2^2-(V^2-w'^2)D_1^2}{D_1^2D_2^2}.$$

After some reduction this becomes

$$\frac{d\lambda}{\lambda}=\frac{U(V^2-w^2)PQ}{D_1^2D_2^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\left.\begin{aligned}
&P=V\{V^2\cos(\chi-\alpha+\theta)-VU\cos\chi+U^2\sin\alpha\cos(\chi+\alpha)\sin(\alpha-\theta)\}\\
&\text{or}\\
&P=(V\cos\theta-u)\{V^2\cos(\chi-\alpha)+v^2\cos(\chi+\alpha)\}\\
&\qquad\qquad-(V\sin\theta-v)\{V^2\sin(\chi-\alpha)+uv\cos(\chi+\alpha)\}\\
&Q=2V^2\cos(2\phi-\chi)-U^2(\cos 2\phi+\cos 2\alpha)\cos\chi\\
&\qquad\qquad+4U^2\sin\alpha\sin\chi\cos\phi\cos(\phi+\alpha)\\
&\text{or}\\
&Q=(2V^2-U^2)\cos(2\phi-\chi)+U^2\{\sin(2\alpha+2\phi)\sin\chi-\cos(2\alpha+\chi)\}
\end{aligned}\right\}\quad(9)$$

where it is to be noticed that Q is independent of the direction of the incident light and P of the inclination of the plate. If the source is fixed in space, $\theta=0$. If it is fixed to the apparatus, $V\sin\theta-v=0$, and

$$\frac{d\lambda}{\lambda}=\frac{U(V^2-w^2)(V\cos\theta-u)\{V^2\cos(\chi-\alpha)+v^2\cos(\chi+\alpha)\}Q}{D_1{}^2D_2{}^2}\,.\quad(10)$$

8. *The Angles.*

$$A=\phi_1{}',$$
$$B=\tfrac{1}{2}\pi-(\phi+\phi_2{}').$$

By § 5 and writing $[x]$ for $e^{x\iota}$,

$$e^{\frac{\phi_1{}'\iota}{2}}=\frac{V\left[\frac{\phi_1}{2}\right]-v\left[-\frac{\phi_1}{2}\right]}{D_v}$$

But $$\phi_1=\phi+\phi'-\frac{\pi}{2};$$

hence

$$\frac{A\iota}{2}=\frac{\left[\frac{\phi}{2}-\frac{\pi}{4}\right]V\left\{V\left[\frac{\phi+\theta}{2}\right]+w\left[-\frac{\phi+\theta}{2}\right]\right\}-v\left[\frac{\pi}{4}-\frac{\phi}{2}\right]\left\{V\left[-\frac{\phi+\theta}{2}\right]+w\left[\frac{\phi+\theta}{2}\right]\right\}}{D_vD_w}$$

$$=\frac{V^2\left[-\frac{\pi}{4}+\phi+\frac{\theta}{2}\right]+V\left\{w\left[-\frac{\pi}{4}-\frac{\theta}{2}\right]-v\left[\frac{\pi}{4}-\phi-\frac{\theta}{2}\right]\right\}-vw\left[\frac{\pi}{4}+\frac{\theta}{2}\right].}{D_1}\quad(11)$$

Again,

$$e^{\frac{\phi_2' \iota}{2}} = \frac{\mathrm{V}\left[\frac{\phi_2}{2}\right] - w\left[-\frac{\phi_2}{2}\right]}{\mathrm{D}_{-w}}$$

and $\phi_2 = \phi - \chi - \chi'$;

hence

$$e^{\frac{\mathrm{B}\iota}{2}} = e^{\left(\frac{\pi}{4} - \frac{\phi}{2} - \frac{\phi_2'}{2}\right)\iota}$$

$$= \left[\frac{\pi}{4} - \frac{\phi}{2}\right] \frac{\left[-\frac{\phi-\chi}{2}\right]\mathrm{V}\left\{\mathrm{V}\left[\frac{\chi+\theta}{2}\right] + w'\left[-\frac{\chi+\theta}{2}\right]\right\} - w\left[\frac{\phi-\chi}{2}\right]\left\{\mathrm{V}\left[-\frac{\chi+\theta}{2}\right] + w'\left[\frac{\chi+\theta}{2}\right]\right\}}{\mathrm{D}_{-w}\mathrm{D}_{w'}}$$

$$= \frac{\mathrm{V}^2\left[\frac{\pi}{4} - \phi + \chi + \frac{\theta}{2}\right] - \mathrm{V}\left\{-w'\left[\frac{\pi}{4} - \phi - \frac{\theta}{2}\right] + w\left[\frac{\pi}{4} - \chi - \frac{\theta}{2}\right]\right\} - ww'\left[\frac{\pi}{4} + \frac{\theta}{2}\right].}{\mathrm{D}_2} \quad . \quad . \quad . \quad (12)$$

Hence after multiplication and reduction

$$\mathrm{D}_1\mathrm{D}_2 e^{\frac{\mathrm{A}-\mathrm{B}}{2}\iota}$$

$$= \mathrm{V}^4\left[-\frac{\pi}{2} + 2\phi - \chi\right]$$

$$+ \mathrm{V}^3\left\{w\left[-\frac{\pi}{2} + \phi - \chi - \theta\right] - w\left[-\frac{\pi}{2} + \phi + \chi + \theta\right] - v[-\chi-\theta] + w'\left[-\frac{\pi}{2} + 2\phi + \theta\right]\right\}$$

$$+ \mathrm{V}^2\left\{-vw[\phi-\chi] + vw[\chi-\phi] - w'v[0] - w^2\left[-\frac{\pi}{2} + \chi\right]\right\}$$

$$+ \mathrm{V}\left\{w^2v[\chi+\theta] + ww'v[-\phi-\theta] - ww'v[\phi+\theta] - w'w^2\left[-\frac{\pi}{2} - \theta\right]\right\}$$

$$+ vw^2w'[0]. \quad . \quad . \quad . \quad (13)$$

Therefore

$$D_1D_2\sin\frac{A-B}{2}=$$

$$-V^4\cos(2\phi-\chi)+V^3\{-2w\sin\phi\sin(\chi+\theta)+v\sin(\chi+\theta)-w'\cos(2\phi+\theta)\}$$
$$+V^2w\{w\cos\chi-2v\sin(\phi-\chi)\}$$
$$+Vw\{-2vw'\sin(\phi+\theta)+vw\sin(\chi+\theta)+ww'\cos\theta\}.$$

After reduction this is found to give

$$\sin\frac{A-B}{2}=-\frac{1}{2}\frac{V\{V-U\cos(\theta-\alpha)\}Q}{D_1D_2}, \quad . \quad . \quad . \quad (14)$$

where Q has the value given above.

When the source is fixed to the apparatus ($V\sin\theta=v$)

$$V\{V-U\cos(\theta-\alpha)\}=V^2-Vu\cos\theta-v^2=V^2\cos^2\theta-Vu\cos\theta$$
$$=V\cos\theta(V\cos\theta-u),$$

and then

$$\sin\frac{A-B}{2}=-\frac{1}{2}\frac{V\cos\theta(V\cos\theta-u)Q}{D_1D_2}.$$

This gives the angle between the wave-fronts.

9. *The distance between the maximal lines is*

$$p=\frac{\lambda_1\lambda_2}{\sqrt{\{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\cos(A-B)\}}}$$

$$=\frac{\lambda_1\lambda_2}{\sqrt{\left\{(\lambda_1-\lambda_2)^2+4\lambda_1\lambda_2\sin^2\frac{A-B}{2}\right\}}}$$

$$=\lambda\frac{(V^2-w^2)^2(V^2-v_4^2)(V^2-w'^2)}{(V^2-w^2)\sqrt{\{U^2P^2Q^2+(V^2-v^2)(V^2-w'^2)V^2(V-U\cos\overline{\theta-\alpha})^2Q^2\}}}$$

$$=\frac{\lambda}{Q}\cdot\frac{(V^2-v^2)(V^2-w'^2)(V^2-w^2)}{\sqrt{\{U^2P^2+V^2(V^2-v^2)(V^2-w'^2)(V-U\cos\overline{\theta-\alpha})^2\}}}; \quad . \quad . \quad . \quad (15)$$

for source fixed to apparatus

$$p=\frac{\lambda}{Q}\frac{(V^2-v^2)(V^2-w'^2)(V^2-w^2)}{(V\cos\theta-u)\sqrt{\{(V^2\cos\overline{\chi-\alpha}+v^2\cos\overline{\chi+\alpha})^2U^2+(V^2-v^2)^2(V^2-w'^2)\}}}$$

$$=\frac{\Lambda\cos\theta}{VQ}\cdot\frac{(V^2-v^2)(V^2-w'^2)(V^2-w^2)}{\sqrt{\{(V^2\cos\overline{\chi-\alpha}+v^2\cos\overline{\chi+\alpha})^2U^2+(V^2-v^2)^2(V^2-w'^2)\}}},$$

where Λ is actual wave-length of light emitted by the source.

The denominator may be simplified so that in this case

$$p=\frac{\Lambda\cos\theta}{V^2Q}\;\frac{(V^2-v^2)(V^2-w^2)(V^2-w'^2)}{\sqrt{[(V^2-v^2)^2\cos^2\chi+\{(V^2-u^2)\sin\chi+2uv\cos\chi\}^2]}}$$

10. This seems the proper place to give a formal proof of the statement made above from general considerations that the apparent frequencies of each wave-train with respect to a point moving with the apparatus are the same.

The velocity of such a point relative to the first train is

$$V+U\sin(\alpha-A).$$

Its apparent frequency is therefore

$$\frac{V+U\sin(\alpha-A)}{\lambda_1}.$$

Now $A=\phi_1'$. Hence

$$V+U\sin(\alpha-A)=V+v\cos\phi_1'-u\sin\phi_1'.$$

Substituting for ϕ_1' in terms of ϕ_1, it will be seen that

$$V+U\sin(\alpha-A)=\frac{V^2-v^2}{D_v}\{V-v\cos\phi_1-u\sin\phi_1\}.$$

Again, $\phi_1=\phi+\phi'-\frac{1}{2}\pi$. Substituting this and expressing ϕ' in terms of its angle of incidence $\phi+\theta$,

$$V-v\cos\phi_1-u\sin\phi_1$$
$$=\frac{V^2-w^2}{D_w}\{V+w\cos(\phi+\theta)-(v\cos\phi+u\sin\phi)\sin(\phi+\theta)\};$$

whence

$$V+U\sin(\alpha-A)=\frac{(V^2-v^2)(V^2-w^2)}{D_1^2}(V-v\sin\theta-u\cos\theta)$$
$$=\frac{\lambda_1}{\lambda}\{V-U\cos(\theta-\alpha)\}.$$

Similarly, it can be shown that

$$V+U\sin(\alpha-B)=\frac{\lambda_2}{\lambda}\{V-U\cos(\theta-\alpha)\};$$

whence

$$\frac{V+U\sin(\alpha-A)}{\lambda_1}=\frac{V+U\sin(\alpha-B)}{\lambda_2}=\frac{V-U\cos(\theta-\alpha)}{\lambda},\quad(16)$$

which shows that the frequencies are equal.

But further,

$$V-U\cos(\theta-\alpha)=V-u\cos\theta-v\sin\theta.$$

Hence when the source is fixed to the apparatus, or $v=\mathrm{V}\sin\theta$,

$$\mathrm{V}-\mathrm{U}\cos(\theta-\alpha)=(\mathrm{V}\cos\theta-u)\cos\theta;$$

and the apparent frequency becomes

$$\frac{(\mathrm{V}\cos\theta-u)\cos\theta}{\lambda}=\frac{\mathrm{V}}{\Lambda};$$

or the apparent frequency is the same as if the whole apparatus were at rest. Whatever, therefore, be the velocity of drift, the apparent frequency remains the same.

11. *Direction of Maximal Lines.*—Let the common direction of the maximal lines make an angle ψ with the first mirror. Then, considering a mesh of the network formed by the two trains of waves, the sides are inclined to the diagonal at angles $180-(\psi+\mathrm{A})$ and $\psi+\mathrm{B}$, supposing $\mathrm{A}>\mathrm{B}$. The sides are proportional to the sines of these angles and also to the wave-lengths. Consequently,

$$\frac{\sin(\psi+\mathrm{A})}{\sin(\psi+\mathrm{B})}=\frac{\lambda_1}{\lambda_2}$$

$$=\frac{\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A})}{\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{B})};$$

whence

$$\tan\psi=\frac{\mathrm{V}\cos\frac{\mathrm{A}+\mathrm{B}}{2}+v\cos\frac{\mathrm{A}-\mathrm{B}}{2}}{\mathrm{V}\sin\frac{\mathrm{A}+\mathrm{B}}{2}-u\cos\frac{\mathrm{A}-\mathrm{B}}{2}}.$$

From the value of $e^{\frac{\mathrm{A}-\mathrm{B}}{2}\iota}$ above,

$$\mathrm{D}_1\mathrm{D}_2\cos\frac{\mathrm{A}-\mathrm{B}}{2}=\mathrm{V}^4\sin(2\phi-\chi)+\mathrm{V}^3\{-2w\cos\phi\sin(\chi+\theta)$$
$$-v\cos(\chi+\theta)+w'\sin(2\phi+\theta)\}$$
$$+\mathrm{V}^2(-w'v-w^2\sin\chi)+\mathrm{V}\{w'w^2\sin\theta+vw^2\cos(\chi+\theta)\}+vw^2w'.$$

Also

$$\mathrm{D}_1\mathrm{D}_2e^{\frac{\mathrm{A}+\mathrm{B}}{2}\iota}$$
$$=\mathrm{V}^4[\chi+\theta]+\mathrm{V}^3\left\{w[-\phi+\chi]-w[\phi-\chi]-v\left[\frac{\pi}{2}-2\phi+\chi\right]+w'[0]\right\}$$
$$+\mathrm{V}^2\left\{vw\left[\frac{\pi}{2}-\phi-\chi-\theta\right]-vw\left[\frac{\pi}{2}-\phi+\chi+\theta\right]+ww[-\phi-\theta]\right.$$
$$\left.-ww'[\phi+\theta]-vw'\left[\frac{\pi}{2}-2\phi-\theta\right]-w^2[-\chi-\theta]\right\}$$
$$+\mathrm{V}\left\{vw^2\left[\frac{\pi}{2}-\chi\right]-w'w^2[0]\right\}+vw^2w'\left[\frac{\pi}{2}+\theta\right].$$

Hence

$$D_1D_2 \sin\frac{A+B}{2} = V^4 \sin(\chi+\theta) + V^3\{-2w\sin(\phi-\chi) - v\cos(2\phi-\chi)\}$$
$$+ V^2\{-2vw\sin\phi\sin(\chi+\theta) - 2ww'\sin(\phi+\theta)$$
$$- vw'\cos(2\phi+\theta) + w^2\sin(\chi+\theta)\}$$
$$+ Vvw^2\cos\chi + vw^2w'\cos\theta;$$

$$D_1D_2\cos\frac{A+B}{2} = V^4\cos(\chi+\theta) + V^3\{w' - v\sin(2\phi-\chi)\}$$
$$+ V^2\{2vw\cos\phi\sin(\chi+\theta) - vw'\sin(2\phi+\theta)$$
$$- w^2\cos(\chi+\theta)\}$$
$$+ V\{vw^2\sin\chi - w'w^2\} - vw^2w'\sin\theta.$$

From these it can be shown, after easy reduction, that

$$D_1D_2\left\{V\cos\frac{A+B}{2} + v\cos\frac{A-B}{2}\right\}$$
$$= (V^2-v^2)(V^2-w^2)\{V\cos(\chi+\theta) + w'\}$$
$$= (V^2-v^2)(V^2-w^2)\{(V\cos\theta-u)\cos\chi - (V\sin\theta-v)\sin\chi\}$$

$$D_1D_2\left\{V\sin\frac{A+B}{2} - u\cos\frac{A-B}{2}\right\}$$
$$= (V^2-w^2)\big[(V\cos\theta-u)\{(V^2-v^2)\sin\chi + 2uv\cos\chi\}$$
$$+ (V\sin\theta-v)(V^2-u^2)\cos\chi\big].$$

Hence

$$\tan\psi = \frac{(V^2-v^2)\{(V\cos\theta-u)\cos\chi - (V\sin\theta-v)\sin\chi\}}{(V\cos\theta-u)\{(V^2-v^2)\sin\chi + 2uv\cos\chi\} + (V\sin\theta-v)(V^2-u^2)\cos\chi} \qquad (17)$$

When the source is fixed to the apparatus ($V\sin\theta = v$),

$$\cot\psi = \tan\chi + \frac{2uv}{V^2-v^2}.$$

That is, if the ratio U/V is very small,

$$\psi = \frac{\pi}{2} - \chi;$$

so that the maximals make for all directions of drift the same angle with the first mirror as the second does, but oppositely inclined.

12. *Position of Central Maximal.*—Let the central maximal

(fig. 5) cut the plane of the first mirror in W, and denote O_1W by x. Draw through O_1 lines parallel to the respective wave-fronts.

The phase of that belonging to the first set at O_1 is the same as that of the incident at O_1, say Π.

The phase of wave-front through W

$$=\Pi-\frac{x \sin A}{\lambda_1}.$$

The phase of the wave-front through O_2 of the second set is the same as that of the incident at O_2, that is of the incident at $O_1+\frac{O_1O_2 \sin(\phi+\theta)}{\lambda}=\Pi+\frac{a \sin(\phi+\theta)}{\lambda}$, since the incident ray makes $\frac{1}{2}\pi-\phi-\theta$ with O_1O_2.

The wave-front through O_1 belonging to this set has phase

$$\Pi+\frac{a \sin(\phi+\theta)}{\lambda}+\frac{O_1M}{\lambda_2}=\Pi+a\left\{\frac{\sin(\phi+\theta)}{\lambda}+\frac{\cos(\phi+B)}{\lambda_2}\right\},$$

and phase of that through W

$$=\Pi+a\left\{\frac{\sin(\phi+\theta)}{\lambda}+\frac{\cos(\phi+B)}{\lambda_2}\right\}-\frac{x \sin B}{\lambda_2}.$$

Since W is on the central maximal these phases are equal. Therefore

$$x\left\{\frac{\sin B}{\lambda_2}-\frac{\sin A}{\lambda_1}\right\}=a\left\{\frac{\cos(\phi+B)}{\lambda_2}+\frac{\sin(\phi+\theta)}{\lambda}\right\}.$$

Now $\lambda_1:\lambda_2=V+U\sin(\alpha-A):V+U\sin(\alpha-B)$.

Hence

$$x[\{V+U\sin(\alpha-A)\}\sin B-\{V+V\sin(\alpha-B)\}\sin A]$$
$$=a\{V+U\sin(\alpha-A)\}\left\{\cos(\phi+B)+\frac{(V^2-w^2)(V^2-w'^2)}{D_2{}^2}\sin(\phi+\theta)\right\},$$

$$2x\sin\frac{B-A}{2}\left\{V\cos\frac{A+B}{2}+v\cos\frac{A-B}{2}\right\}$$
$$=\frac{(V^2-v^2)(V^2-w^2)}{D_1{}^2D_2{}^2}\{V-U\cos(\theta-\alpha)\}\{D_2{}^2\cos(\phi+B)$$
$$+(V^2-w^2)(V^2-w'^2)\sin(\phi+\theta)\}.$$

Substituting the values for the factors on the left already obtained, and dividing out,

$$xV\{V\cos(\chi+\theta)+w'\}Q=a\{D_2^2\cos(\phi+B)$$
$$+(V^2-w^2)(V^2-w'^2)\sin(\phi+\theta)\}.$$

From the value of $e^{\frac{Bi}{2}}$ in § 8 it easily follows that

$$D_2^2e^{(\phi+B)\iota}=V^4\left[\frac{\pi}{2}-\phi+2\chi+\theta\right]+V^3\left\{2w'\left[\frac{\pi}{2}-\phi+\chi\right]-2w\left[\frac{\pi}{2}\right]\right\}$$
$$+V^2\left\{-2ww'\left[\frac{\pi}{2}+\chi+\theta\right]-2ww'\left[\frac{\pi}{2}-\chi-\theta\right]\right.$$
$$\left.+w'^2\left[\frac{\pi}{2}-\phi-\theta\right]+w^2\left[\frac{\pi}{2}+\phi-2\chi-\theta\right]\right\}$$
$$+V\left\{-2ww'^2\left[\frac{\pi}{2}\right]+2w^2w'\left[\frac{\pi}{2}+\phi-\chi\right]\right\}+w^2w'^2\left[\frac{\pi}{2}+\phi+\theta\right];$$

whence after a short reduction,

$$D_2^2\cos(\phi+B)+(V^2-w^2)(V^2-w'^2)\sin(\phi+\theta)$$
$$=2V(V^2-w^2)\{V\cos(\chi+\theta)+w'\}\sin(\phi-\chi).$$

Therefore

$$x=2a\frac{V^2-w^2}{Q}\sin(\phi-\chi). \quad . \quad . \quad . \quad . \quad (18)$$

Since this is independent of θ, *the position of the central point is the same whether the source is fixed in the æther or to the apparatus.*

13. We have now obtained the magnitudes of the various quantities required for the discussion of the experiment. For this purpose, however, it will be convenient to express them in terms of quantities defining the configuration of the apparatus. Let **C** denote the angle between the two mirrors, and let **A**, **B** denote the angles which the first and second mirrors respectively make with the plate. Then

$$\mathbf{C}=\tfrac{1}{2}\pi-\chi,$$
$$\mathbf{A}=\tfrac{1}{2}\pi-\phi,$$
$$\mathbf{B}=\phi-\chi.$$

Further, denote the distance of O_1 from the plane of the second mirror by b. Then

$$b=a\sin\mathbf{B}=a\sin(\phi-\chi).$$

It will be convenient to collect the formulæ expressed in these new quantities:—

$$\frac{\lambda_1-\lambda_2}{\lambda}=\frac{U(V^2-w^2)PQ}{D_1{}^2D_2{}^2},$$

$$\sin\tfrac{1}{2}(B-A)=\frac{1}{2}\,\frac{V\{V-U\cos(\theta-\alpha)\}}{D_1D_2}Q,$$

$$p=\frac{\lambda}{Q}\cdot\frac{(V^2-v^2)(V^2-w^2)(V^2-w'^2)}{\sqrt{\{U^2P^2+V^2(V^2-v^2)(V^2-w'^2)(V-U\cos\overline{\theta-\alpha})^2\}}},$$

$$\tan\psi=\frac{(V^2-v^2)\{(V\cos\theta-u)\sin\mathbf{C}-(V\sin\theta-v)\cos\mathbf{C}\}}{(V\cos\theta-u)\{V^2-v^2)\cos\mathbf{C}+2uv\sin\mathbf{C}\}+(V\sin\theta-v)(V^2-u^2)\sin\mathbf{C}}.$$

$$x=2b\,\frac{V^2-w^2}{Q};$$

where

$$P=V\{V_2\sin(\mathbf{C}+\alpha-\theta)-VU\sin\mathbf{C}+U^2\sin\alpha\sin(\alpha-\theta)\sin(\mathbf{C}-\alpha)\}$$
$$=(V\cos\theta-u)\{V^2\sin(\mathbf{C}+\alpha)+v^2\sin(\mathbf{C}-\alpha)\}$$
$$-(V\sin\theta-v)\{V^2\cos(\mathbf{C}+\alpha)+uv\sin(\mathbf{C}-\alpha)\},$$
$$Q=2V^2\sin(\mathbf{B}-\mathbf{A})-U^2\{\sin(\mathbf{B}-\mathbf{A})+\sin(2\alpha-2\mathbf{A})\cos\mathbf{C}+\sin(\mathbf{C}-2\alpha)\}.$$

For the case of $V\sin\theta=v$, source fixed to apparatus,

$$\sin\tfrac{1}{2}(B-A)=\frac{1}{2}\,\frac{V\cos\theta(V\cos\theta-u)}{D_1D_2}Q,$$

$$p=\frac{\Lambda\cos\theta}{V^2Q}\,\frac{(V^2-v^2)(V^2-w^2)(V^2-w'^2)}{\sqrt{[(V^2-v^2)^2\sin^2\mathbf{C}+\{(V^2-u^2)\cos\mathbf{C}+2uv\sin\mathbf{C}\}^2]}}$$

$$\cot\psi=\cot\mathbf{C}+\frac{U^2}{V^2-v^2}\sin 2\alpha,$$

$$P=(V\cos\theta-u)\{V^2\sin(\mathbf{C}+\alpha)+v^2\sin(\mathbf{C}-\alpha)\}.$$

When there is no motion,

$$\lambda_1=\lambda_2=\lambda,$$
$$\sin\tfrac{1}{2}(B-A)=\sin(\mathbf{B}-\mathbf{A}),$$
$$p=\frac{\lambda}{2\sin(\mathbf{B}-\mathbf{A})},$$
$$\psi=\mathbf{C},$$
$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})}.$$

If h be measured in wave-lengths $=n\lambda$ say,

$$x=\frac{n\lambda}{\sin(\mathbf{B}-\mathbf{A})}=2np;$$

or there are twice as many bands from O_1 to the central band as there are wave-lengths in O_1L.

When there is motion Q has the same value as if the apparatus were at rest when the direction of drift is given by

$$\sin(\mathbf{B}-\mathbf{A})+\sin(2\alpha-2\mathbf{A})\cos\mathbf{C}+\sin(\mathbf{C}-2\alpha)=0, \quad . \quad (19)$$

which gives two real values for $\tan\alpha$. In the actual case $\mathbf{C}$ is nearly 90° and $\mathbf{B}$ nearly $=\mathbf{A}$; whence α is nearly $\frac{\pi}{4}$ or $\pi+\frac{\pi}{4}$, i. e. *the drift produces no effect when it is nearly parallel to the plane of the plate.*

The interfering waves have the same frequencies when either

(1) $U=0$,
(2) $w=\pm V$,
(3) $Q=0$,
(4) $P=0$.

(1) is the case of no motion.

(2) makes the velocity of the plate perpendicular to itself equal to that of light—a case which may be put aside.

(3) is the case which will have to be discussed immediately.

(4) gives the direction of drift to be that of the maximal lines. It is the case, indeed, already discovered by general reasoning in § 3. To prove the statement it is necessary to show that when $\alpha=\pi-\psi$, $P=0$. It may be sufficient to do this for the case of a fixed source. The equations are:—

$$-\cot\alpha=\cot\mathbf{C}+\frac{2uv}{V^2-v^2},$$

$$P=(V\cos\theta-u)\{V^2\sin(\mathbf{C}+\alpha)+v^2\sin(\mathbf{C}-\alpha)\}$$
$$=(V\cos\theta-u)\{(V^2+v^2)\sin\mathbf{C}\cos\alpha+(V^2-v^2)\cos\mathbf{C}\sin\alpha\}.$$

Now the first equation gives at once

$$(V^2-v^2)\cos\mathbf{C}\sin\alpha+\cos\alpha\sin\mathbf{C}(V^2-v^2)+2uv\sin\mathbf{C}\sin\alpha=0,$$

and
$$u\sin\alpha=v\cos\alpha,$$

which makes the second factor of P vanish.

14. The preceding formulæ hold whatever the velocity of drift may be. It is, however, extremely unlikely that this

velocity amounts to anything approximating to 100 times the velocity of the earth in its orbit, which latter is roughly 10^{-4} times that of light. Even in this extreme case the square of the ratio of the velocities may be neglected in all terms except those which are divided by a small quantity. In the discussion which follows we shall suppose the conditions to be those of the actual apparatus of Michelson and Morley, and that the square of the velocity ratios may be neglected except under the circumstances mentioned above. The ratio U/V will be denoted by ξ.

The breadth of a band is then given by

$$p=\frac{\Lambda}{2\sin(\mathbf{B}-\mathbf{A})-\xi^2\{\sin(\mathbf{B}-\mathbf{A})+\sin(2\alpha-2\mathbf{A})\cos\mathbf{C}+\sin(\mathbf{C}-2\alpha)\}}.$$

In the particular apparatus **A**, **B** were nearly 45° and **C** nearly 90°. Hence in terms which multiply ξ^2 we may put $\mathbf{A}=\mathbf{B}=45^\circ$, $\mathbf{C}=90^\circ$ exactly. Then

$$p=\frac{\Lambda}{2\sin(\mathbf{B}-\mathbf{A})-\xi^2\cos 2\alpha}.$$

If $\mathbf{B}-\mathbf{A}$ is $<\frac{1}{2}\xi^2$, the breadths of the bands of a fringe will change from the breadth they would have without drift to infinity, as the direction of the drift alters.

In the case of $\xi=10^{-4}$, $\Lambda=5\,.\,10^{-5}$ cm., and $\mathbf{B}-\mathbf{A}=\frac{1}{2}\xi^2$, the breadth of a band when the drift is parallel to the plate is 50 metres; so that such a disposition would be impossible to observe. If the velocity of drift were 100 times that of the earth's orbit, this minimum breadth would be 5 mm., and under these circumstances the observations would have shown enormous variations in the breadth of the bands.

It may be taken as certain then that $\mathbf{B}-\mathbf{A}$ was considerably larger than ξ^2, although in itself exceedingly small. If c denote the breadth of band when there is no drift or when $\alpha=\frac{\pi}{4}$, then in general

$$p=\frac{c}{1-\dfrac{\xi^2}{2\sin(\mathbf{B}-\mathbf{A})}\cos 2\alpha}.$$

It has been seen that for all probable values of ξ, $\sin(\mathbf{B}-\mathbf{A})$ will be considerably greater than $\frac{1}{2}\xi^2$, so that $\xi^2/2\sin(\mathbf{B}-\mathbf{A})$ will be a small quantity; whence

$$p=c\left\{1+\frac{1}{2}\frac{\xi^2\cos 2\alpha}{\sin(\mathbf{B}-\mathbf{A})}\right\}.$$

Also to this order $\psi = C$, or the direction of the maximals is constant.

15. It remains to determine the magnitude of the effect referred to in § 4. By means of the lenses of the eye, aided or not by an optical instrument, an image is thrown on the retina of points on a certain plane on which the instrument is focussed. Of any such point P an image is formed at Q (say) on the retina. Through P two rays (belonging to the two systems of waves) pass, which by the optical apparatus are again brought to pass through Q, having passed over the same optical distance D′, say. The waves travel with the same velocity but have different wave-lengths. Suppose at P the phase-difference is Π, *i. e.* the λ_2 phase greater by Π than the λ_1 phase. Then in the wave diagram the two paths (D′) are occupied by waves of different lengths λ_1, λ_2, and therefore at Q the phase-difference of λ_2 over λ_1 is $\Pi + \frac{D'}{\lambda_2} - \frac{D'}{\lambda_1}$, or there is an increased difference of phase of amount $D'\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$. Consequently the resulting intensity at Q will not be the same as that at P—the maxima and minima on the retina are not the images of the maxima and minima of the object viewed. Now the λ_2 waves are ahead of the λ_1 on the right hand of the maximals. Hence in order that Q may be the central bright band it must be the optical image of a point on the left of the centre of the original fringe, for at Q the λ_2 waves lead still more than at P, and the distance from it must be $D'\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$ bands.

Now by (8)

$$\frac{\lambda_1 - \lambda_2}{\lambda} = \frac{U(V^2 - w^2)P \, . \, Q}{D_1^2 D_2^2},$$

$$\frac{\lambda_1 \lambda_2}{\lambda^2} = \frac{(V^2 - w^2)^2 (V^2 - v^2)(V^2 - w'^2)}{D_1^2 D_2^2};$$

$$\therefore \quad D' \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} = \frac{D'}{\lambda} \cdot \frac{UPQ}{(V^2 - w^2)(V^2 - v^2)(V^2 - w'^2)}$$

$$= \frac{D'(V\cos\theta - u)}{\lambda} \cdot \frac{U\cos\alpha \, . \, Q}{V^4}$$

$$= \frac{D'}{\Lambda} \frac{Q}{V^2} \xi \cos\alpha$$

$$= \frac{D'}{p} \xi \cos\alpha ;$$

and $D'\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}$ bands $=D'\xi\cos\alpha$ cm.

Hence the distance of the observed central fringe is

$$x=D+\frac{D\xi^2}{\Lambda}c\cos 2\alpha+D'\xi\cos\alpha. \quad . \quad . \quad . \quad (20)$$

In the actual experiment D was about 11 metres, and, so far as can be estimated from the description, D′ nearly 3 metres.

The relative importance of these terms depends for given ξ on the value chosen for the breadth of the band. No indication is given of this, but it must probably have been adjusted to something between $\frac{1}{10}$ mm. and 10 mm. The following are a few cases, taking $\Lambda=5\,.\,10^{-5}$ cm. and writing $x=a+b\cos 2\alpha+c\cos\alpha$, and measuring in millims.,

	Band	·1	1	10
$\xi=10^{-4}$ Earth's orbital velocity.	b	·2	2	20
	c	·3	·3	·3
$\xi=10^{-5}$.	b	·002	·02	·2
	c	·03	·03	·03

For $\xi=10^{-4}$ a breadth of ·1 mm. would therefore make both terms equally sensible; whilst for $\xi=10^{-5}$, the second would overpower the first, but would be of equal weight for a breadth of 1 mm. If $D'=D$, *i. e.* if the instrument was focussed on the actual first mirror, then for $\xi=10^{-5}$ a breadth of band of 1 cm. would make both terms about equally sensible.

Finally, aberration would displace the apparent position to the right. Hence

$$x=a+b\cos 2\alpha+c\cos\alpha-d\sin\alpha,$$

where d involves the first power of ξ.

16. From conversation with Professor Morley I learn that in the actual experiment no slit was used, and that the reading-telescope was focussed on the first mirror. To complete the theory therefore it is necessary to take account of the finite breadth of the flame. The flame may be regarded as built up of vertical narrow strips of light, from each of which, as has been seen, there results a system of fringes on whatever point the reading-telescope is focussed. It is necessary then to see whether these fringes overlap, or whether there are places where the superposed fringes exactly fit. Now the fringes are given by the maximals; and the

question is, Are there *loci* where the maximals for all the various component slits coincide?

The general formulæ developed above involve the angle θ, —the angle which the incident light makes with the first mirror. If a point on the flame makes an angle θ with the datum line, it will be sufficient to replace θ by $\theta_1+\theta$ in the various formulæ in order to obtain the result for this part of the flame.

The direction of drift now makes an angle $\alpha-\theta_1$ with the line from the slit to the centre of the plate. Hence

$$\begin{aligned}\mathrm{V}\sin\theta&=\mathrm{U}\sin(\alpha-\theta_1)\\&=v\cos\theta_1-u\sin\theta_1.\end{aligned}$$

The position of the white point—*i. e.* where the central maximal cuts the first mirror, is given by

$$x=2b\,.\,\frac{\mathrm{V}^2-w^2}{\mathrm{Q}},$$

θ_1 only enters through w. Hence when ξ is small x is independent of θ_1—that is, the central maximal cuts the plane of the first mirror in a point which is the same for all points of the plane.

Next the common direction of the maximals is given by (17). In this we put $\theta_1+\theta$ for θ. Now

$$\mathrm{V}\cos(\theta+\theta_1)-u=\mathrm{V}\cos\theta\cos\theta_1-\mathrm{V}\sin\theta_1\sin\theta-u,$$

where the source is fixed to the apparatus,

$$\mathrm{V}\sin\theta=v\cos\theta_1-u\sin\theta_1;$$

whence

$$\begin{aligned}\mathrm{V}\cos(\theta_1+\theta)-u&=\cos\theta_1\{\mathrm{V}\cos\theta-u\cos\theta_1-v\sin\theta_1\},\\\mathrm{V}\sin(\theta_1+\theta)-v&=\sin\theta_1\{\mathrm{V}\cos\theta-u\cos\theta_1-v\sin\theta_1\}.\end{aligned}$$

Hence

$$\cot\psi=\frac{\mathrm{V}^2\sin(\theta_1+\chi)-v^2\sin\chi\cos\theta_1+2uv\cos\chi\cos\theta_1-u^2\cos\chi\sin\theta_1}{(\mathrm{V}^2-v^2)\cos(\theta_1+\chi)},$$

or

$$\begin{aligned}\cot\psi&=\tan(\chi+\theta_1)+\frac{v^2\sin\theta_1+2uv\cos\theta_1-u^2\sin\theta_1}{\mathrm{V}^2-v^2}\cdot\frac{\cos\chi}{\cos(\chi+\theta_1)}\\&=\tan(\chi+\theta_1)+\frac{\mathrm{U}^2\sin(2\alpha-\theta_1)}{\mathrm{V}^2-v^2}\cdot\frac{\cos\chi}{\cos(\chi+\theta_1)}.\end{aligned}$$

When ξ^2 is small

$$\psi=\tfrac{1}{2}\pi-(\chi+\theta_1).$$

The maximals for different small strips of the flame are therefore differently inclined. It has been seen that these intersections with the plane of the first mirror coincide. Hence they coincide nowhere else. Consequently a fringe will be seen if the telescope is focussed on the first mirror, which will gradually become more and more indistinct as the plane on which it is focussed recedes more and more from it.

If γ denote the angular breadth of the flame as seen from the point on the plate where the datum line meets it, the maximals for the various points of the flame will form a system of pencils of angular breadth γ, whose vertices pass through the maximal points on the first mirror which have just been shown to be the same for every point on the flame. The fringe on a screen parallel to the first mirror will then completely fade into white light when its distance from the first mirror is such that the pencils intersect, each, the succeeding one. This takes place at a distance y such that $y\gamma$ = breadth of a band. At distances greater than y no fringes can be seen at all. At distances nearly as great as y we should expect dark lines on a more or less uniformly bright background.

Discussion of the Displacement of Fringe.

17. The position of the central band is given by

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-\frac{1}{2}\xi^2\cos 2\alpha}.$$

If then $\mathbf{B}>\mathbf{A}$, b must be positive, that is the plane of the second mirror must lie to the right of the intersection of the first mirror and the plate. If on the other hand $\mathbf{A}>\mathbf{B}$, b must be negative, or the second plane must lie to the left of the same intersection. It will be convenient to distinguish these two cases as the B and the A type respectively.

Suppose an experiment to start with the drift in the same direction as the incident light. Then as the drift alters from this position in either direction, the central band is displaced to the right in the B type and to the left in the A type. As $\mathbf{A}-\mathbf{B}$ is exceedingly small—of order 10^{-5} (or 2 sec.) at most—the adjustment of the mirrors can easily change from one type to the other on consecutive days. It follows that averaging the results of different days in the usual manner is not allowable unless the types are all the same. If this is not attended to the average displacement may be expected to come out zero—at least if a large number are averaged.

If D denote the distance of the central band from the intersection of the plate and mirrors, when there is no drift—or when the drift is parallel to the plate—

$$D = \frac{b}{\sin(\mathbf{B}-\mathbf{A})}$$

$$x = \frac{D\sin(\mathbf{B}-\mathbf{A})}{\sin(\mathbf{B}-\mathbf{A}) - \frac{1}{2}\xi^2\cos 2\alpha}$$

$$\left.\begin{array}{r}\text{Displacement to}\\ \text{left of this position}\end{array}\right\} = x - D = \frac{\frac{1}{2}\xi^2 D\cos 2\alpha}{\sin(\mathbf{B}-\mathbf{A}) - \frac{1}{2}\xi^2\cos 2\alpha}.$$

In any given case suppose $\sin(\mathbf{B}-\mathbf{A}) = \frac{1}{2}k\xi^2$, then

$$\text{Displacement} = \frac{D\cos 2\alpha}{k - \cos 2\alpha}.$$

Fig. 6.

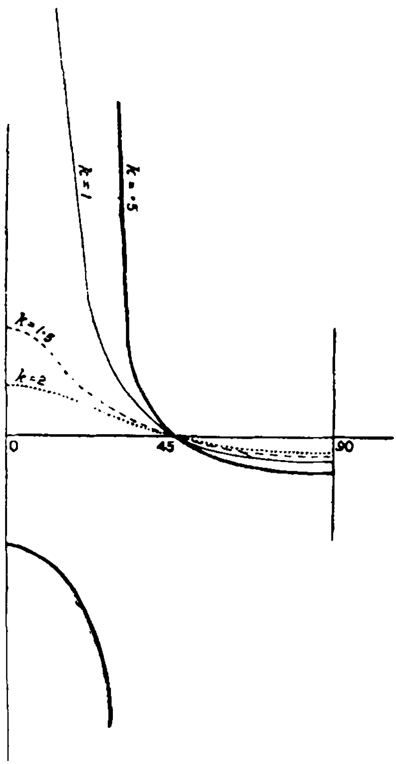


Fig. 6 shows how the displacement changes with α, for the four cases of $k = \cdot 5$, 1, 1·5, and 2. When $k < 1$, these curves have an asymptote which as k increases from 0 to 1 moves up to the origin. The negative infinity branch becomes narrower and at the same time moves off to infinity. When $k = 1$ there is no negative branch, but a series of curves with positive infinite peaks. As k increases beyond 1, these peaks shorten until when k is large the curves become the ordinary harmonic curve $y = k \cos 2\alpha$. In Michelson and Morley's experiment k was apparently always large.

Discussion of Michelson and Morley's Observations.

18. The result of §17, that it is not allowable to average the results of different sets of observations until the type of each has been determined, naturally leads us to a reconsideration of the numerical data obtained by Michelson and Morley, who did lump together the observations taken on different days. I propose to show that, instead of giving a null result, the numerical data published in their paper show distinct evidence of an effect of the kind to be expected.

It may here be recalled that in taking an observation, the apparatus was rotated in its mercury bath and readings taken at 16 equidistant points as the reading-telescope passed them. On each occasion this was repeated six times, and the means of the six readings in each position taken. These means are the numbers printed in their paper. They are given for noon on July 8, 9, and 11, and for 6 P.M. on July 8, 9, and 12. The means of these three days are taken and then the means of the first eight and of the second eight, thus eliminating any effect depending on $\cos \alpha$ alone. The result is that there is no apparent displacement of the fringe.

In looking at the sets of readings, one is struck at once with the fact that all the readings continuously increase or decrease. This is evidently the effect of temperature changes. For short intervals, it is extremely likely that the temperature disturbances will be a linear function of the time. If this is exactly so, and if the readings were taken at equal intervals of time, it is possible to eliminate the disturbances due to this. For the readings at the beginning and at the end of a complete revolution ought (in absence of temperature effects) to be the same, whilst on the supposition made above there would be a temperature error altering by equal steps for each successive reading—in a way to be indicated immediately. The readings for each set of complete revolutions should first be corrected in this way and then the average of the six taken to eliminate accidental and personal errors. I have applied this correction to the published means. The result is the same as if the correction had first been applied to each series and the results then averaged. This would have been preferable if possible, as a comparison of each set with the average would have given data for a measure of the probable error.

To illustrate the method of applying this correction, I give here the working of the 6 P.M. observations on July 8 :—

	I.	61·2,	63·3,	63·3*,	68·2,	67·7,	69·3,	70·3,	69·8,	69·0
	II.	61·2,	62·1,	63·0,	63·9,	64·8,	65·7,	66·6,	67·5,	68·4
	III.	0	+1·2	+·3	+4·3	+2·9	+3·6	+3·7	+2·3	+·6
	I.	69·0,	71·3,	71·3,	70·5,	71·2,	71·2,	70·5,	72·5,	75·7
	II.	68·4,	69·3,	70·2,	71·1,	72·0,	72·9,	73·8,	74·7,	75·7
	III.	+·6	+2·0	+1·1	−·6	−·8	−1·7	−3·3	−2·2	+0
		0	+1·2	+·3	+4·3	+2·9	+3·6	+3·7	+2·3	+·6
	IV.	·6	+3·2	+1·4	+3·7	+2·1	+1·9	+·4	+·1	+·6
Deduct 1·7.	V.	−1·1	+1·5	−·3	+2·0	+·4	+·2	−1·3	−1·6	−1·1

In the above, line I. gives the published observations. Line II. gives a series of numbers, increasing by equal steps from 61·2 to 75·7. Line III. gives the differences of I. & II., and therefore, according to our supposition, freed from the temperature errors. Line IV. gives the mean of the first eight and the last eight, and therefore eliminates effects depending on cos α (see § 15). Line V. is the result obtained by deducting 1·7 from each, so as to reduce readings to give deviations from the mean, 1·7=sum/8.

The other sets are treated in the same way. The results are as follows:—

Noon Observations.

July 8. −1·0+0·2+1·2+0·7+2·8−1·1+0·2+2·6−1·0
July 9. −1·7−1·8−0·2+0·3+0·4+1·1+2·2−1·4−1·7
July 11. 0·9−2·2−2·1−2·7+0·3+2·0+1·9+1·7+0·9

P.M. Observations.

July 8. −1·1+1·5−0·3+2·0+0·4+0·2−1·3−1·6−1·1
July 9. −1·3−0·7+0·1−0·5+1·1+1·6+0·3−0·6−1·3
July 12. 0·4−1·0−2·2−1·6−2·0+1·4+3·3+1·5+0·4

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* If this were a MS. misreading for 65·3, the correct numbers in lines III., IV., and V. would be 2·3, 3·4, and 1·7, which, as a glance at the curve in Plate I. will show, would make the curve regular.

These results are represented graphically on Plate I. All the curves give distinct evidence of a cos 2α effect, except noon of July 8, and possibly of 6 P.M. of July 8; the latter certainly if the supposition of the footnote is correct. Moreover, the curves clearly show that the observations of July 9, 11, 12 belong to one type (B or A), and those of July 8 to the other (A or B). The last curve represents the average of all, on this supposition. That is the value of the ordinates are one-third of July 9 + July 11 − July 8 and July 9 + July 12 − July 8.

The evidence is also strengthened by comparison of the noon and P.M. curves. The drift at 6 P.M would be at right-angles to that at noon, consequently we should expect the curve to be shifted half a period (*i. e.* 90°) with reference to the first. Now in the P.M. observations, the rotation of the apparatus was in the opposite direction to that at noon. Consequently, in order to compare the curves, the numbers for the P.M. curves should be read backwards. This is done in the dotted curves in Pl. I. A glance at the curves renders evident the fact that the shift shown is of the right character. The amplitude of the P.M. curve should be less than that of the noon in the ratio $\sin^2\lambda$* where λ is the latitude.

The preceding attempt to get rid of the temperature effect is not proposed as one which gives an accurate result. The object is to show that the observations of Michelson and Morley do give an affirmative answer to the question "Is there a drift of æther past the earth?" The argument is sufficient to show that the experiments should be repeated with extreme care to eliminate temperature errors, and if possible *in vacuo*. If possible the absence of observers would be desirable, and, for the reason stated in § 15, also the absence of reading-telescopes. We have seen that if a slit is used for the light source, fringes are formed on a screen placed in any position. This points to a method in which a photographic film on a rotating drum is exposed automatically.

The FitzGerald-Lorentz Effect.

19. Amongst the various explanations advanced to account for the supposed null result of Michelson and Morley's experiment, the best known and accepted is that first proposed, I believe, by G. F. FitzGerald, viz., that the very motion of a

* That is supposing the component of the drift perpendicular to the plane of the apparatus to produce no effect. It probably does produce an effect, however.

solid through the æther produces a small extension perpendicular to the direction of drift, or contraction in the direction, the amount being proportional to ξ^2. This has received some justification from the fact that on the theories of Larmor and of Lorentz as to the connexion between matter and æther, a contraction such as is indicated should be expected. At first sight—especially in the theory of the experiment as given by Michelson and Morley—an effect of this kind of the proper amount might seem capable of annulling any observed displacement. If, however, the effect of such a contraction on the displacement of the fringe be worked out on the lines of the rigorous theory developed in the present paper, it will be found that not only is it incapable of explaining the null result, but that in fact it should increase the displacement to be observed. To produce annulment an extension along the line of drift is required. It is the object of the present section to prove this and to show how by a suitable modification of Michelson and Morley's experiment it lies at our disposal to test the truth or otherwise of Larmor and Lorentz's result, and, if such contraction exists, to measure its amount.

We shall assume then that when a solid moves through the æther, it suffers a contraction along the direction of the drift such that a length l parallel to it is contracted by an amount $k\xi^2 l$. The effective changes in the apparatus are those which occur in the horizontal plane. If then the total drift makes an angle I with the plane of the apparatus, lines in the plane will be distorted as if there were a contraction $k\xi^2 \cos \mathrm{I} . l$ in the direction of the component of the drift in the plane, or—if ξ represent this component as in the preceding pages—a contraction $k\xi^2 l/\cos \mathrm{I}$. In what immediately follows we shall replace $k/\cos \mathrm{I}$ by k.

20. Owing to the distortion produced both the lengths of lines and the magnitudes of angles lying in the plane will be altered. It is first necessary to determine this alteration.

In fig. 7 (p. 40) AB denotes any line, LB the direction of drift. Draw AL perpendicular to BL.

Through the contraction B comes to B′ where

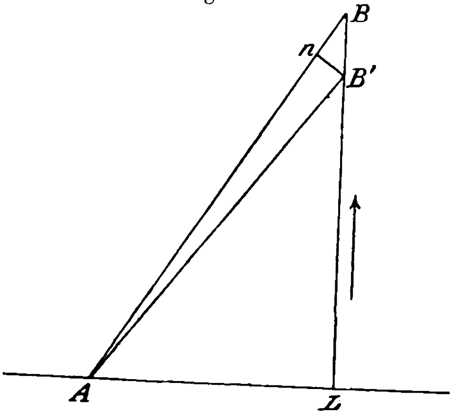
$$\mathrm{BB'} = k\xi^2 . \mathrm{BL}$$

and the line AB is displaced to AB′. If AB $= r$, ABL $= \theta$,

$$\begin{aligned}\delta r = \mathrm{B}n &= k\xi^2 . \mathrm{BL} \cos \mathrm{ABL} \\ &= k\xi^2 r \cos^2 \theta = \tfrac{1}{2} k\xi^2 r (1 + \cos 2\theta), \\ \delta\theta = \mathrm{BAB'} &= \frac{\mathrm{B'}n}{r} = \frac{k\xi^2 . \mathrm{BL} \sin \theta}{r} \\ &= \tfrac{1}{2} k\xi^2 \sin 2\theta.\end{aligned}$$

These formulæ hold only when $k\xi^2$ is small. Hence the following results are not applicable to cases where the velocity

Fig. 7.



of drift is comparable with that of light. Further, we shall suppose ξ^2 so small that it may be neglected in comparison with unity. In this case we write V^2 for V^2-w^2 and

$$x=\frac{2b\sin\mathbf{C}}{Q}$$

$$=\frac{b\sin\mathbf{C}}{\sin(\mathbf{B}-\mathbf{A})-\frac{1}{2}\xi^2 . R},$$

where
$$R=(1-\cos\mathbf{C})\sin(\mathbf{C}-2\alpha)$$
$$=\cos 2\alpha \text{ when } \mathbf{C}=90^\circ.$$

The change in $b\sin\mathbf{C}$ will be of order ξ^2 and may be neglected in comparison with it. So, too, the change in ξ^2R will be of order ξ^4 and may be neglected. There remains only the change in $\mathbf{B}-\mathbf{A}$.

In fig. 8, O_1X, O_2Z denote the directions of the planes of the mirrors and O_1Y that of the plate. XYZ is any line perpendicular to the direction of drift. Then to compute the deformation we have

$$\mathbf{A}=XO_1Y=XO_1L-YO_1L,$$
$$\delta\mathbf{A}=\delta . XO_1L-\delta . YO_1L$$
$$=\tfrac{1}{2}k\xi^2\{\sin 2\alpha-\sin 2(\alpha-\mathbf{A})\}.$$

So also

$$\delta\mathbf{B}=\tfrac{1}{2}k\xi^2\{\sin 2(\alpha-\mathbf{A})+\sin 2(\mathbf{C}-\alpha)\};$$

$$\therefore\quad \delta(\mathbf{B}-\mathbf{A})=\tfrac{1}{2}k\xi^2\{\sin 2(\mathbf{C}-\alpha)+2\sin 2(\alpha-\mathbf{A})-\sin 2\alpha\}.$$

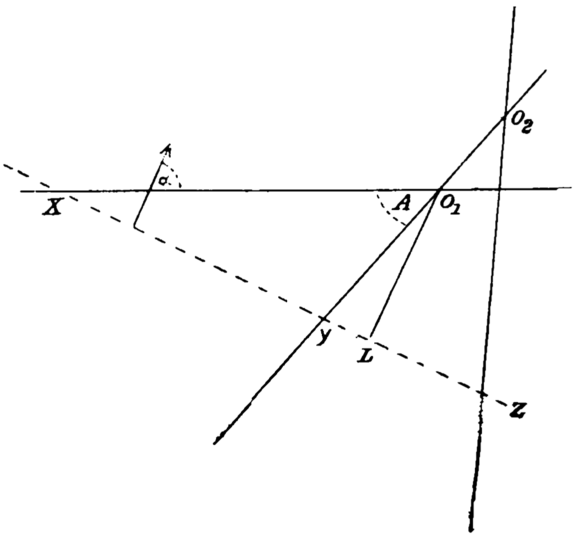
If any fringes are to be seen at all, $\mathbf{B}-\mathbf{A}$ must be exceedingly small. Hence in the small term multiplied by ξ^2 we may put $\mathbf{B}=\mathbf{A}$, $\mathbf{C}=2\mathbf{A}$. Then it can be shown that

$$\delta(\mathbf{B}-\mathbf{A})=-k\xi^2(1-\cos\mathbf{C})\sin(\mathbf{C}-2\alpha)=-k\xi^2\mathrm{R};$$

and, since $\cos(\mathbf{B}-\mathbf{A})=1$,

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-(k+\frac{1}{2})\mathrm{R}\xi^2}.$$

Fig. 8.



21. In Michelson and Morley's experiment $\mathbf{C}=90^\circ$, $k=\cos 2\alpha$,

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-(k+\frac{1}{2})\xi^2\cos 2\alpha}.$$

Hence to annul the effect k should be $-\frac{1}{2}$, and consequently an expansion instead of a contraction is necessary.

If now λ denote the true coefficient of contraction due to drift $k=\lambda/\cos\mathrm{I}$, and if the vertical component of drift produce (owing to motion of source) no direct effect in displacing the fringe

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-(\lambda/\cos\mathrm{I}+\frac{1}{2})\xi^2\cos 2\alpha}.$$

Observations at noon and at 6 P.M. give the direction of the projection of the total drift on the plane of the apparatus —that is, we have the projections of the same direction on two known planes. These planes are the position of the horizontal at noon and the same plane turned 90° round the axis of the earth. Consequently the absolute *direction* of drift is determined, and the values of I for noon and 6 P.M. can be calculated—say I_1, I_2.

The magnitudes of the shift of the fringes then give

$$(\lambda/\cos I_1 + \tfrac{1}{2})\Xi^2 \cos^2 I_1 \quad \text{and} \quad (\lambda/\cos I_2 + \tfrac{1}{2})\Xi^2 \cos^2 I_2,$$

where Ξ is the ratio of total drift to velocity of light. Hence λ and Ξ can be separately determined.

III. *The Decomposition of Hydrogen Peroxide by Light, and the Electrical Discharging Action of this Decomposition.* By R. F. D'Arcy, *M.A.**

THE progress of knowledge on the subject of the probable modes of development of electrical phenomena in the atmosphere during the last few years has been rapid. The difference in behaviour of positive and negative ions as nuclei of condensation, combined with the proved existence of ions as being normally present in the air, may be considered as establishing the precipitation theory of atmospheric electricity.

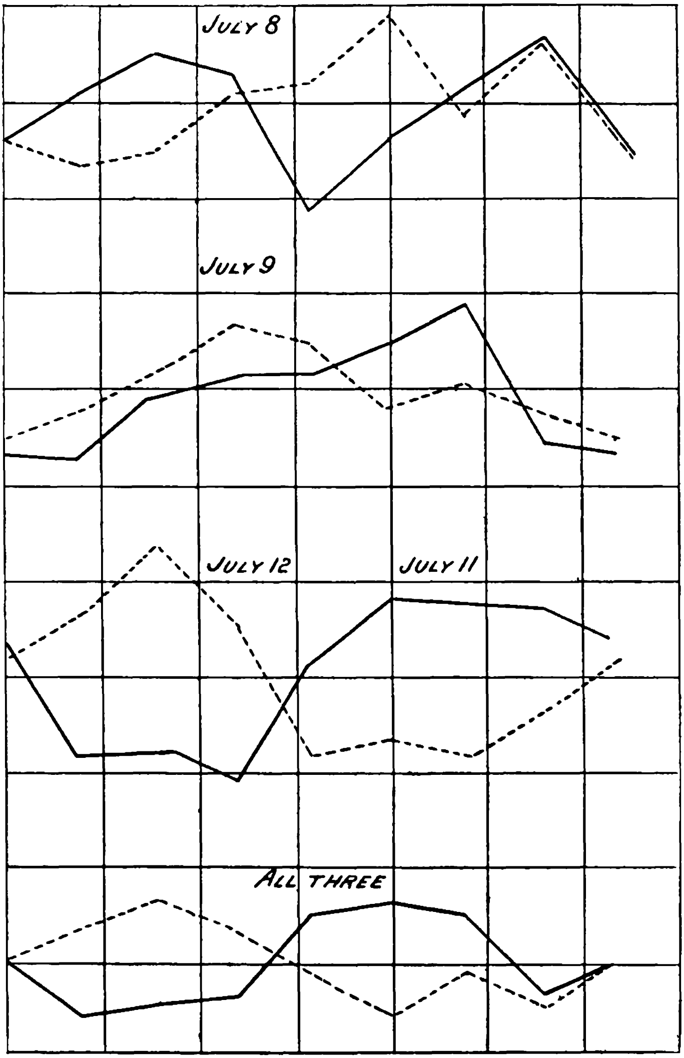
The object of the present paper is to give an account of experiments which were undertaken with the view of investigating a chemical action to account for a possible origin of these charged particles. It may be stated that the view adopted is that such a possible action is the decomposition of hydrogen peroxide *by light*†. The water formed by the decomposition being positively charged, and the oxygen (whatever its atomicity may be) being negatively charged.

The formation of hydrogen peroxide in nature is a well recognized fact. The probable formation of hydrogen peroxide by ultra-violet light in moist oxygen has been indicated by C. T. R. Wilson (Phil. Trans. 1899); the minute drops of hydrogen-peroxide solution being uncharged. The main ideas of the present paper are to suggest that hydrogen peroxide *when split up* yields two parts which are oppositely charged, and that in nature this splitting up may be brought about by the action of light.

The experiments made were of two kinds. Some were made to investigate the conditions under which hydrogen peroxide is decomposed by light, as, although this effect has

* Communicated by the Author.

† The possible formation of ions by supersaturation has been considered by C. T. R. Wilson, Phil. Trans. vol. cxciii. A. 1900.


JULY 8
JULY 9
JULY 12
JULY 11
ALL THREE